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Asymptotic modelling of Skin-effects in coaxial cables

Geoffrey Beck · Sébastien Imperiale ·
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Abstract In this work we tackle the modeling of non-perfectly conducting thin coaxial cables. From the non-dimensionnalised 3D Maxwells equations, we derive, by asymptotic analysis with respect to the (small) transverse dimension of the cable, a simplified effective 1D model and an effective reconstruction procedure of the electric and magnetic fields. The derived effective model involves a fractional time derivatives that accounts for the so-called skin effects in highly conducting regions.

Keywords Maxwell's equation · Asymptotic analysis · Coaxial cables

1 Introduction

The present paper continues the efforts undertaken in [6, 9, 5] which aimed at the 1D modeling of electric cables with a view to application to non-destructive testing [3, 11]. A coaxial cable is a tube made up of dielectric material surrounding a metallic core and being surrounded by a metallic screen. Such a cable is usually modelled, in the engineering literature [17], using a 1D linear wave type equation that describes the propagation of current and voltage: the telegrapher's equations. This equation can be represented as a succession of infinitesimal RLCG quadripoles. In [6, 9], such models were derived from an asymptotic analysis of Maxwell's 3D equations when the transverse dimension of the cable tends to 0, in the case where the metal parts are perfect conductors. In these derived models, the RLCG coefficients can be calculated explicitly by knowing the geometry and the physical parameters of the cable (permittivity, permeability, electrical conductivity, magnetic conductivity). In these models, the resistance coefficient R is

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proportional to the magnetic conductivity. However, in practice such models are not very satisfactory for the description of realistic attenuation phenomena because the magnetic conductivity is excessively low (or even zero). However, there exists another cause of losses related to the so called skin effect linked to the fact that, in reality, the metallic parts of the cable are not perfectly conducting which authorizes the electromagnetic field to slightly penetrate them. This work aims at taking into account this effect in a 1D effective model. The skin effect is due to the high contrast of electrical conductivity between the dielectric part (inside which it is small) and the metallic part of the cable (inside which it is very large). It is also characterized by the presence of thin layers in the conductive parts which adjoin the dielectric part in which all the electromagnetic energy of the conductors is concentrated. A dimensional analysis suggests that two small parameters must be taken into account: the cable diameter and the skin depth (distance of penetration of a monochromatic wave in a conductive medium). The new model will be derived from Maxwell's equations by a multi-scale asymptotic analysis in which the above parameters are related to a single one denoted δ: the diameter of the cable is proportional to δ and the skin depth proportional to δ^2. This allows for an asymptotic analysis with respect to δ. At order 0 of this analysis, one recovers the usual telegrapher's equations without losses (i.e. without the resistance coefficient R and conductance coefficient G). These coefficients will only appear at order 1 in the model. As we shall see, in the frequency domain, the resistance coefficient will appear multiplied by $\sqrt{i\omega}$, which corresponds in time domain to a convolution.

The outline of the paper is as follows. Section 2 is devoted to a description of the problem under study. After a geometrical description of co-axial cables (Section 2.1), we present Section 2.2 the governing equations. Through a classical rescaling of the space and time variables, we propose (Section 2.3) a nondimensionalized version of these equations posed in a reference cable, which allows to identify several small parameters. We then relate these parameters to a single one $\delta > 0$, that roughly corresponds to the diameter of the cable and formulate (Section 2.4) the δ-dependent family of problems (14), which are posed in a fixed geometry but involve δ-dependent differential operators, that will be the subject of our asymptotic analysis for small δ.

In Section 3, we give the main results of this work. We recall first (Section 3.1) some basic mathematical material, most of which is borrowed from our previous works, that are used to define the resistance, conductance, capacitance and inductance coefficients. These coefficients are then used to define in Section 3.2 the 1D effective model.

Section 4 is devoted to a presentation of an Ansatz for the asymptotic expansion in δ of the solution of the equations obtained in Section 2. This is based on a reformulation of the problems as a transmission problem between the dielectric parts and the conducting parts of the cable (Section 4.2) and an additional change of variable of boundary layer type in the conducting parts of the cable (Section 4.3) that relies on appropriate local coordinates close to the interfaces (defined Section 4.1).

The construction of the effective model is the subject of Section 5, which is the main section of the paper. The results of this construction are used in Section 6 for proposing an 1D effective model (Section 6.1), the stability of which is proven in

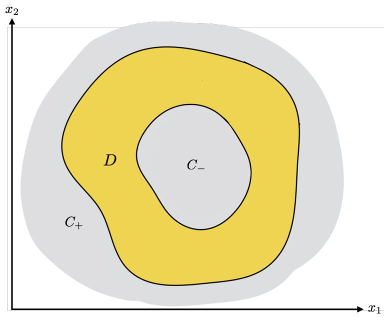

Fig. 1 Section of a coaxial cable. C_+ and C_- are the outer and inner conducting domain of the dielectric D.

Section 6.2 and the 1D model is completed by 3D reconstruction formulae (Section 6.3) for the fields inside the dielectric. Finally, in section 7, we pay a particular attention to the case of circular symmetric cables, for which we can check that our analysis permits to recover and justify some results of the electrical engineering literature. The analysis that we present is formal but we are pretty convinced that the use of techniques similar to [10] and [8,16] would allow us to derive error estimates.

2 Mathematical formulation of the problem

2.1 Geometry of the cables

We consider the propagation of electromagnetic waves in an open domain Ω that is the disjoint union of a conducting domain Ω_c and a dielectric one Ω_d,

$$\overline{\Omega} = \overline{\Omega_c} \cup \overline{\Omega_d},$$

where Ω_c and Ω_d are cylindrical domains given by

$$\Omega_c = C \times \mathbb{R}, \quad \Omega_d = D \times \mathbb{R},$$

where $C = C_+ \cup C_-$, C_+ corresponding to the outer metallic shield and C_- to the inner metallic wire and D is non-simply connected, see Figure 1.

2.2 Maxwell's equations in cables

The propagation of electric field $\mathbf{E}(\mathbf{x},t) = (\mathbf{E}_1,\mathbf{E}_2,\mathbf{E}_3)^t(\mathbf{x},t)$ and magnetic field $\mathbf{H}(\mathbf{x},t) = (\mathbf{H}_1,\mathbf{H}_2,\mathbf{H}_3)^t(\mathbf{x},t)$ in these domains is governed by Maxwell's equations

$$\begin{cases} \varepsilon\, \partial_t \mathbf{E} + \sigma\, \mathbf{E} - \boldsymbol{\nabla} \times \mathbf{H} = \mathbf{j}, \\ \mu\, \partial_t \mathbf{H} + \boldsymbol{\nabla} \times \mathbf{E} = \mathbf{0}. \end{cases} \tag{1}$$

The equations are set in the whole space $\mathbf{x} = (x_1,x_2,x_3) \in \mathbb{R}^3$ and for positive time t. The physical parameters involved are the permittivity $\varepsilon(\mathbf{x})$, the permeability $\mu(\mathbf{x})$, the electric conductivity $\sigma(\mathbf{x})$ and $\mathbf{j}(\mathbf{x},t) = (j_1,j_2,j_3)^t(\mathbf{x},t)$ which stands for a source term. Let us recall that, provided that the source term $\mathbf{j}$ is divergence free, then the fields satisfy the following free divergence conditions.

$$\boldsymbol{\nabla} \cdot (\varepsilon\, \partial_t \mathbf{E} + \sigma\, \mathbf{E}) = 0, \quad \boldsymbol{\nabla} \cdot (\mu \partial_t \mathbf{H}) = 0. \tag{2}$$

To describe the propagative behavior observed in industrial applications it appears judicious to introduce the tangential components

$$\mathbf{E}_T = (\mathbf{E}_1, \mathbf{E}_2)^t, \quad \mathbf{H}_T = (\mathbf{H}_1, \mathbf{H}_2)^t,$$

of the electric and magnetic fields. We can rewrite the equations (1) with these new unknowns. To do so, we shall use, for all scalar functions u and 2D vector fields $\mathbf{v}$ with two components v_1 and v_2, the following notations,

$$\nabla u \equiv \begin{pmatrix} \partial_1 u \\ \partial_2 u \end{pmatrix}, \quad \mathbf{rot}\; u \equiv \begin{pmatrix} \partial_2 u \\ -\partial_1 u \end{pmatrix}, \quad \mathrm{div}\; \mathbf{v} \equiv \partial_1 v_1 + \partial_2 v_2, \quad \mathrm{rot}\; \mathbf{v} \equiv \partial_1 v_2 - \partial_2 v_1,$$

and, for any $\mathbf{v} = (v_1, v_2)$ and $\mathbf{u} = (u_1, u_2)$,

$$\mathbf{v} \cdot \mathbf{u} \equiv v_1\, u_1 + v_2\, u_2, \qquad \mathbf{v} \times \mathbf{u} \equiv v_1\, u_2 - v_2\, u_1, \qquad \mathbf{e}_3 \times \mathbf{v} \equiv (-v_2, v_1)^t.$$

Then equations (1) are set in $\Omega \times \mathbb{R}^+$ and can be recast as

$$\begin{cases} \varepsilon\, \partial_t \mathbf{E}_T + \sigma\, \mathbf{E}_T - \partial_3\, (\mathbf{e}_3 \times \mathbf{H}_T) - \mathbf{rot}\; \mathbf{H}_3 = (j_1, j_2)^t, \\ \varepsilon\, \partial_t \mathbf{E}_3 + \sigma\, \mathbf{E}_3 - \mathrm{rot}\; \mathbf{H}_T = j_3, \\ \mu\, \partial_t \mathbf{H}_T + \partial_3\, (\mathbf{e}_3 \times \mathbf{E}_T) + \mathbf{rot}\; \mathbf{E}_3 = \mathbf{0}, \\ \mu\, \partial_t \mathbf{H}_3 + \mathrm{rot}\; \mathbf{E}_T = 0. \end{cases} \tag{3}$$

Because we assume later that the conductivity in the conducting domain is very high, the electric fields are strongly damped. Therefore, the nature of the boundary conditions that is employed on $\partial\Omega$ will not play any specific role as long as it provides a well-defined problem. Denoting $\mathbf{n} = (n^t, 0)^t$ the outward unitary of Ω, we arbitrarily choose to impose

$$\mathbf{E} \times \mathbf{n} = \mathbf{0} \quad \text{on} \quad \partial\Omega \quad \Rightarrow \quad \mathbf{E}_T \times n = \mathbf{0} \quad \text{and} \quad \mathbf{E}_3 = 0 \quad \text{on} \quad \partial\Omega.$$

Finally, for simplicity we assume that the system is at rest at $t = 0$, more precisely

$$\mathbf{E}(\mathbf{x}, 0) = \mathbf{H}(\mathbf{x}, 0) = \mathbf{0}. \tag{4}$$

2.3 Nondimensionalization

The nondimensionalization of the equations will enable us to explain the regime of the phenomena in which we want to focus on. We define by

- λ a typical wavelength in the longitudinal direction,
- d the mean diameter of its cross-sections.

This leads us to introduce the dimensionless space coordinates variables

$$\widehat{\mathbf{x}} = S(\mathbf{x}) := \Big(\frac{x_1}{d}, \frac{x_2}{d}, \frac{x_3}{\lambda}\Big)^t \tag{5}$$

and for any set $O \subset \mathbb{R}^3$, we defined the rescaled domain $\widehat{O}$ by

$$\widehat{O} := \big\{\widehat{\mathbf{x}} = S(\mathbf{x}) \;/\; \mathbf{x} \in O\big\}. \tag{6}$$

We define accordingly the rescaled relative permittivity and permeability as

$$\forall\, \widehat{\mathbf{x}} \in \widehat{\Omega}, \quad \varepsilon_r(\widehat{\mathbf{x}}) = \varepsilon(\mathbf{x})/\varepsilon_0, \quad \mu_r(\widehat{\mathbf{x}}) = \mu(\mathbf{x})/\mu_0, \quad \widehat{\mathbf{x}} = S(\mathbf{x}). \tag{7}$$

We define by T a characteristic period of a plane-wave with wavelength λ,

$$T = \lambda/c_0, \quad c_0 := (\varepsilon_0\,\mu_0)^{-\frac{1}{2}},$$

so that we can introduce a dimensionless time

$$\widehat{t} = t/T \equiv c_0\, t/\lambda. \tag{8}$$

Finally, exploiting – see the first two equations of (3) – that the conductivity σ is homogeneous to ε_0 divided by a time, we introduce the dimensionless rescaled conductivity

$$\forall\, \widehat{\mathbf{x}} \in \widehat{\Omega}, \quad \sigma_r(\widehat{\mathbf{x}}) = \sigma(\mathbf{x})/\sigma_0, \quad \sigma_0 := \varepsilon_0/T = \frac{1}{\lambda}\sqrt{\frac{\varepsilon_0}{\mu_0}}, \quad \widehat{\mathbf{x}} = S(\mathbf{x}). \tag{9}$$

Let us introduce $E^\star > 0$ and $H^\star > 0$ two quantities respectively homogeneous to an electric and magnetic field. We use them to rescale the fields $\mathbf{E}(x,t)$ and $\mathbf{H}(x,t)$ into dimensionless fields $E(\widehat{\mathbf{x}},\widehat{t}\,)$ and $H(\widehat{\mathbf{x}},\widehat{t}\,)$ as:

$$E(\widehat{\mathbf{x}},\widehat{t}\,) := \mathbf{E}(\mathbf{x},t)/E^\star, \quad H(\widehat{\mathbf{x}},\widehat{t}\,) := \mathbf{H}(\mathbf{x},t)/H^\star, \quad \widehat{\mathbf{x}} = S(\mathbf{x}), \;\; \widehat{t} = t/T. \tag{10}$$

We impose that the scaling quantities $E^\star > 0$ and $H^\star > 0$ satisfy

$$E^\star\sqrt{\varepsilon_0} = H^\star\sqrt{\mu_0}.$$

With some elementary algebra, one finds that the new unknowns (E,H) satisfy the following system set in $\widehat{\Omega} \times \mathbb{R}^+$, with $\mathbf{j}_T := (j_1, j_2)^t$:

$$\begin{cases} \varepsilon_r\,\partial_t E_T + \sigma_r\, E_T - \partial_3\,(\mathbf{e}_3 \times H_T) - \dfrac{\lambda}{d}\,\mathbf{rot}\; H_3 = \mathbf{j}_T, \\[2ex] \varepsilon_r\,\partial_t E_3 + \sigma_r\, E_3 - \dfrac{\lambda}{d}\,\mathrm{rot}\; H_T = j_3, \\[2ex] \mu_r\,\partial_t H_T + \partial_3\,(\mathbf{e}_3 \times E_T) + \dfrac{\lambda}{d}\,\mathbf{rot}\; E_3 = \mathbf{0}, \\[2ex] \mu_r\,\partial_t H_3 + \dfrac{\lambda}{d}\,\mathrm{rot}\; E_T = 0, \end{cases} \tag{11}$$

where for simplicity, the differential operators in the rescaled variables $(\widehat{\mathbf{x}},\widehat{t}\,)$ are denoted as with the original variables. In (11), $\mathbf{j} = (j_1, j_2, j_3)^t$ is an appropriately rescaled dimensionless source term.

2.4 Identification of a small parameter and parametrized family of problems

Our goal is to obtain a simplified approximate model for the problem (11) by embedding the problem in a family of problems parametrized by a small parameter. As it is always the case in asymptotic modeling, the construction of the family of problems is somewhat arbitrary but aims at taking into account the particular geometrical and physical aspects of the problem:

(i) The cable is thin, i. e., d is small with respect to λ: $d \ll \lambda$,

(ii) The conductivity is small in the domain D,

(iii) The conductivity is very large in the domains $C_{\pm}$.

This would lead to introduce three small parameters: the first one, for taking into account the geometrical criterion (i), being

$$\delta := \frac{d}{\lambda}. \tag{12}$$

However, in order to simplify the analysis, we will relate the other small parameter to δ according them to the following considerations:

- To take into account (ii), we shall assume that inside $\widehat{D}$, the (dimensionless) conductivity σ_r is taken proportional to δ.
- Taking into account (iii) amounts to model the well-known skin effect that expresses the fact that the electromagnetic field penetrates inside the conducting parts $\widehat{C}_{\pm}$ only inside two thin layers that touch $\widehat{D}$ and which thickness ℓ, referred as the skin depth, is supposed to be very small with respect to the diameter of the cable, i. e., $\ell \ll d$, which is happens to be reasonable in many industrial context. According to (12), this suggests to impose that ℓ/λ is proportional to δ^2

  This will allow us to scale appropriately the conductivity in the domains $\widehat{C}_{\pm}$. Indeed, it is well known [14,15] or [8,7,19] that the skin depth ℓ of a monochromatic wave of frequency $\omega/2\pi > 0$ is related to the permeability μ and to the conductivity σ of the conductor via the formula

  $$\ell^2 \sim (\mu\sigma\omega)^{-1}.$$

  Since $\lambda = 2\pi c_0/\omega$ and using (7) and (9), tedious but straightforward calculations show that the above formula becomes

  $$(\ell/\lambda)^2 \sim \frac{1}{\mu_r \sigma_r}.$$

  Getting ℓ/λ proportional to δ^2 then imposes to take σ_r proportional to δ^{-4} inside $\widehat{C}_{\pm}$.

That is why, for our family of problems parametrized by δ, we shall consider a family of (dimensionless) conductivity distributions $\sigma_r^\delta(\widehat{\mathbf{x}})$ such that

$$\begin{cases} \sigma_r^\delta(\widehat{\mathbf{x}}) := \delta \; \sigma_{r,d}(\widehat{\mathbf{x}}), \quad \widehat{\mathbf{x}} \in \widehat{\Omega}_d = \widehat{D} \times \mathbb{R}, \\[2ex] \sigma_r^\delta(\widehat{\mathbf{x}}) := \delta^{-4} \; \sigma_{r,c}(\widehat{\mathbf{x}}), \quad \widehat{\mathbf{x}} \in \widehat{\Omega}_c = \widehat{C} \times \mathbb{R}, \end{cases} \tag{13}$$

where $\sigma_{r,d}$ is a fixed (i. e. independent of δ) functions defined $\widehat{D}$ and $\sigma_{r,c}(\widehat{\mathbf{x}})$ is constant and strictly positive in each connected component of the conducting domain, more precisely (see however Remark 2 and Remark 4),

$$\sigma_{r,c}(\widehat{\mathbf{x}}) = \sigma_{r,c}^{\pm}, \qquad \widehat{\mathbf{x}} \in \widehat{C}_{\pm} \times \mathbb{R}.$$

We will see that the weak conductivity in the dielectric and the very strong conductivity in the conducting part will result for the electromagnetic field in dissipation of the same order (see Section 3). In the following, we shall denote $\varepsilon_{r,d}(\widehat{\mathbf{x}})$ and $\mu_{r,d}(\widehat{\mathbf{x}})$ the restrictions of $\varepsilon_r(\widehat{\mathbf{x}})$ and $\mu_r(\widehat{\mathbf{x}})$ to the dielectric medium $\widehat{\Omega}_d$. We also assume that the permittivity and permeability $\varepsilon_r(\widehat{\mathbf{x}})$ and $\mu_r(\widehat{\mathbf{x}})$ are constant in the inner and outer conductors $\widehat{\Omega}_c$ (see Remark 2 and Remark 4 again),

$$\varepsilon_{r,c}(\widehat{\mathbf{x}}) = \varepsilon_{r,c}^{\pm}, \qquad \mu_{r,c}(\widehat{\mathbf{x}}) = \mu_{r,c}^{\pm}, \qquad \widehat{\mathbf{x}} \in \widehat{C}_{\pm} \times \mathbb{R}.$$

Finally, we introduced the following parametrized family of problems, deduced from (11): for each $\delta > 0$, find (E^δ, H^δ) solutions, $\widehat{\Omega} \times \mathbb{R}^+$, of

$$\begin{cases} \varepsilon_r\, \partial_t E_T^\delta + \sigma_r^\delta\, E_T^\delta - \partial_3\,(\mathbf{e}_3 \times H_T^\delta) - \dfrac{1}{\delta}\, \mathbf{rot}\; H_3^\delta = \mathbf{j}_T^\delta, \\[2ex] \varepsilon_r\, \partial_t E_3^\delta + \sigma_r^\delta\, E_3^\delta - \dfrac{1}{\delta}\, \mathrm{rot}\; H_T^\delta = j_3^\delta, \\[2ex] \mu_r\, \partial_t H_T^\delta + \partial_3\,(\mathbf{e}_3 \times E_T^\delta) + \dfrac{1}{\delta}\, \mathbf{rot}\; E_3^\delta = \mathbf{0}, \\[2ex] \mu_r\, \partial_t H_3^\delta + \dfrac{1}{\delta}\, \mathrm{rot}\; E_T^\delta = 0. \end{cases} \tag{14}$$

where $\mathbf{j}^\delta = (\mathbf{j}_T^\delta, j_3^\delta)$ is a family of well-prepared source terms in order to be compatible with the free divergence condition. More precisely, the divergence free condition reads in rescaled coordinates: $\mathrm{div}\, \mathbf{j}_T^\delta + \delta\, \partial_3\, j_3^\delta = 0$. Thus, to simplify the analysis, we assume that

$$\mathbf{j}_T^\delta = \mathbf{j}_T^0 + \delta\, \mathbf{j}_T^1 \quad \text{and} \quad j_3^\delta = j_3^0 \tag{15}$$

where $\mathbf{j}_T^0$, $\mathbf{j}_T^1$ and j_3^0 are $\delta-$independent fields supported in $\widehat{D} \times \mathbb{R}$ satisfying

$$\mathrm{div}\, \mathbf{j}_T^0 = 0 \quad \text{and} \quad \mathrm{div}\, \mathbf{j}_T^1 + \partial_3\, j_3^0 = 0. \tag{16}$$

3 Main result

3.1 Preparatory material

This section is mainly a recap of well known mathematical results on 2D vector fields in their particular version developed in [10].

Green's formulae

Denoting by $\mathbf{u} = (u_1, u_2)$ a smooth 2D vector field and by v a smooth scalar field, then the following Green's formulae hold

$$\int_{\widehat{D}} v \;\mathrm{div}\; \mathbf{u}\, \mathrm{d}x = - \int_{\widehat{D}} \mathbf{u} \cdot \nabla\; v\, \mathrm{d}x + \int_{\partial \widehat{D}} \mathbf{u} \cdot n\, v\, \mathrm{d}s \tag{17}$$

and
$$\int_{\widehat{D}} v \ \mathrm{rot}\, \mathbf{u}\, \mathrm{d}x = \int_{\widehat{D}} \mathbf{u} \cdot \mathbf{rot}\ v\, \mathrm{d}x - \int_{\partial \widehat{D}} \mathbf{u} \times n\, v\, \mathrm{d}s \tag{18}$$
where $\partial \widehat{D} = \Sigma_+ \cup \Sigma_-$. As usual, equalities (17) and (18) can be extended to functions $v \in H^1(\widehat{D})$ and $\mathbf{u} \in H(\mathrm{div}, \widehat{D})$, or respectively $\mathbf{u} \in H(\mathrm{rot}, \widehat{D})$, by changing the integral boundary terms into duality parings between $H^{-1/2}(\partial \widehat{D})$ and $H^{1/2}(\partial \widehat{D})$.

The 2D harmonic electric and magnetic potentials φ_e and in ψ_m in $\widehat{D}$.

These two scalar fields are functions of $\mathbf{x}$ and are defined as the solutions of two particular elliptic problems in the (dielectric) section $\widehat{D}$ in which x_3 plays the role of a parameter through the dependence in x_3 of $\varepsilon_{r,d}$ and $\mu_{r,d}$. To ease the notations, in the sequel, we omit to write this dependency.

The field $\varphi_e \in H^1(\widehat{D})$, called electric harmonic potential, is the unique solution of the non homogeneous Dirichlet problem and $\psi_m \in H^1(\widehat{D} \setminus \Gamma)$ satisfies
$$\begin{cases} \mathrm{div}\, \varepsilon_{r,d}\, \nabla \varphi_e = 0, & \text{in } \widehat{D} \\ \varphi_e = 0 & \text{on } \Sigma_+ \\ \varphi_e = 1 & \text{on } \Sigma_-. \end{cases} \tag{19}$$
To characterize the magnetic harmonic potential ψ_m, we need to introduce a cut $\Gamma \subset \widehat{D}$ namely a line that joins the inner boundary Σ_- to the the outer boundary Σ_+ in such a way that the open set $\widehat{D} \setminus \Gamma$ is simply connected. We define a unit normal vector n along Γ. We then define $\psi_m \in H^1(\widehat{D} \setminus \Gamma)$ as the unique solution of the non homogeneous jump problem
$$\begin{cases} \mathrm{div}\, \mu_{r,d}\, \nabla \psi_m = 0 & \text{in } \widehat{D} \setminus \Gamma, \\ \mu_{r,d}\, \nabla \psi_m \cdot n = 0 & \text{on } \partial \widehat{D}, \\ [\psi_m]_\Gamma = 1, \quad [\mu_{r,d}\, \nabla \psi_m \cdot n]_\Gamma = 0, & \text{across } \Gamma, \end{cases} \tag{20}$$
that satisfies the mean value condition
$$\int_{\widehat{D}} \mu_{r,d}\, \psi_m\, \mathrm{d}x = 0. \tag{21}$$
The jump $[u]_\Gamma$ (respectively the normal jump $[\mathbf{v} \cdot n]_\Gamma$) of a function u (respectively a vector field) which is smooth enough except maybe across Γ, is defined by
$$[u]_\Gamma = \gamma_+ u - \gamma_- u, \quad [\mathbf{v} \cdot n]_\Gamma = (\gamma_+ \mathbf{v}) \cdot n - (\gamma_- \mathbf{v}) \cdot n, \tag{22}$$
where we have set
$$\forall\, x \in \Gamma, \quad \gamma_\pm u(x) := \lim_{\eta \to 0^+} u(x \pm \eta\, n). \tag{23}$$
As it is classical, the definition (22) can be extended by density to scalar fields u in $H^1(\widehat{D} \setminus \Gamma)$ and vector fields $\mathbf{v}$ in $H(\mathrm{div}, \widehat{D} \setminus \Gamma)$. Let us point out (cf. [10])

although ψ_m depends on the cut Γ, it is not the case of the field $\widetilde{\nabla}\psi_m \in L^2(\widehat{D})^2$ defined by (note that $\mathcal{D}(\widehat{D} \setminus \Gamma)$ is dense in $L^2(\widehat{D})$)

$$\forall\, \theta \in \mathcal{D}(\widehat{D}_\Gamma)^2, \quad \int_{\widehat{D}} \widetilde{\nabla}\psi_m \cdot \theta\, dx = \langle \nabla\psi_m, \theta\rangle_{\widehat{D}_\Gamma}, \quad \text{where } \widehat{D}_\Gamma := \widehat{D} \setminus \Gamma \tag{24}$$

and $\langle \cdot,\cdot\rangle_{\widehat{D}_\Gamma}$ denotes here the duality bracket between $\mathcal{D}'(\widehat{D}_\Gamma)$ and $\mathcal{D}(\widehat{D}_\Gamma)$.

Note also that from (24) one can deduce that $\operatorname{rot} \widetilde{\nabla}\psi_m = 0$.

The main interest of the fields φ_e and ψ_m lies in the following theorem that is a generalisation to non-homogeneous problems of results given in [1,2,12].

Theorem 1 (Lemmas 3.1 and 3.3 of [9])

(*a*) $\big\{\mathbf{v} \in L^2(\widehat{D})^2 \;/\; \operatorname{rot} \mathbf{v} = 0, \operatorname{div}(\varepsilon_{r,d}\mathbf{v}) = 0, \mathbf{v} \times n_\pm|_{\Sigma_\pm} = 0\big\} = \operatorname{Span}\big[\nabla\varphi_e\big]$,

(*b*) $\big\{\mathbf{v} \in L^2(\widehat{D})^2 \;/\; \operatorname{rot} \mathbf{v} = 0, \operatorname{div}(\mu_{r,d}\mathbf{v}) = 0, \mu_{r,d}\, \mathbf{v} \cdot n_\pm|_{\Sigma_\pm} = 0\big\} = \operatorname{Span}\big[\widetilde{\nabla}\psi_m\big]$.

Finally, a useful property of the potentials is the following (see [9] for a proof),

$$\int_{\widehat{D}} \nabla\varphi_e \cdot\, (\mathbf{e}_3 \times \widetilde{\nabla}\psi_m)\, \mathrm{d}x = -\int_{\widehat{D}} \widetilde{\nabla}\psi_m \cdot (\mathbf{e}_3 \times \nabla\varphi_e)\, \mathrm{d}x = -1. \tag{25}$$

Effective capacitance, inductance, resistance and conductance coefficients

The following effective coefficients, expressed in terms of the potentials φ_e and ψ_m will appear later in the construction of our effective model. More precisely, the effective capacitance and inductance coefficients are given respectively by

$$C = \int_{\widehat{D}} \varepsilon_{r,d}\, |\nabla\varphi_e|^2\, \mathrm{d}x > 0, \quad L = \int_{\widehat{D}} \mu_{r,d}\, |\widetilde{\nabla}\psi_m|^2\, \mathrm{d}x > 0 \tag{26}$$

and the conductance coefficient

$$G = \int_{\widehat{D}} \sigma_{r,d}\, |\nabla\varphi_e|^2\, \mathrm{d}x. \tag{27}$$

Obviously, $\sigma_{r,d} \geq 0$ implies $G \geq 0$ but $G > 0$ as soon as $\sigma_{r,d}$ does not vanish identically in $\widehat{D}$. To be more precise we can state the following theorem.

Theorem 2 *The conductance coefficient G is strictly positive as soon as $\sigma_{r,d}$ is strictly positive in a small open subset of the cross-section.*

Proof We proceed by contradiction. Let $\widehat{D}_+ \subset \widehat{D}$ a connected open subset of $\widehat{D}$ such that $\sigma_{r,d}$ is strictly positive in $\widehat{D}_+$. If G were equal to 0, this would mean the existence of a constant C_e such that $\varphi_e = C_e$ in $\widehat{D}_+$. Since $\varepsilon_{r,d}$ is Lipschitz-continuous we can apply the unique continuation principle (see [22], Section 1.8) to $\varphi_e - C_e$ and deduce that $\varphi_e - C_e$ is identically zero in $\widehat{D}$. However this contradicts the boundary conditions of (19). ■

Note that C, L and G only depend on the dielectric medium and already appeared for instance in the articles [9,10]. A new coefficient is introduced to take into account the presence of the conductor and the skin-effect. In order to define it, we need the following regularity assumption (see Remark 1)

$$\widetilde{\nabla}\psi_m \times n \in L^2(\partial\widehat{D}). \tag{28}$$

Under this assumption the resistance coefficient R is defined by

$$R = \int_{\partial\widehat{D}} \beta_c \left|\widetilde{\nabla}\psi_m \times n\right|^2 \mathrm{d}s, \quad \text{where } \beta_c := \sqrt{\frac{\mu_{r,c}}{\sigma_{r,c}}}. \tag{29}$$

Remark 1 Since the interfaces $\Sigma_\pm$ have C^2 regularity, owing to standard elliptic regularity results (see [20,21]) to ensure (28) it is sufficient that $\mu_{r,d}$ is Lipschitz-continuous in a tubular neighbourhood of both interfaces $\Sigma_\pm$.

The 2D electric and magnetic conjugate potentials φ_m and in ψ_e in $\widehat{D}$.

We will also use later two scalar fields defined as the solutions of elliptic problems similar to (19) and (20). More precisely, we introduce the $\varphi_m \in H^1_0(\widehat{D})$ as the unique solution of the following non homogeneous Dirichlet problem,

$$\begin{cases} \operatorname{div} \mu_{r,d}^{-1} \nabla\varphi_m = 0, & \text{in } \widehat{D} \\ \varphi_m = 0 & \text{on } \Sigma_+ \\ \varphi_m = 1 & \text{on } \Sigma_-. \end{cases} \tag{30}$$

and $\psi_e \in H^1(\widehat{D}\setminus\Gamma)$ as the unique solution of the non homogeneous jump problem,

$$\begin{cases} \operatorname{div} \varepsilon_{r,d}^{-1} \nabla\psi_e = 0 & \text{in } \widehat{D}\setminus\Gamma, \\ \varepsilon_{r,d}^{-1} \nabla\psi_e \cdot n = 0 & \text{on } \partial\widehat{D}, \\ [\psi_e]_\Gamma = 1, \quad [\varepsilon_{r,d}^{-1} \nabla\psi_e \cdot n]_\Gamma = 0, & \text{across } \Gamma, \end{cases} \tag{31}$$

that satisfies the same mean value condition (21), i. e.,

$$\int_{\widehat{D}} \mu_{r,d}\, \psi_e \,\mathrm{d}x = 0. \tag{32}$$

The following relations between the potentials and the corresponding conjugate potentials were shown in [10]

$$(a)\ \ \varepsilon_{r,d}\nabla\varphi_e = C\, \widetilde{\mathbf{rot}}\, \psi_e, \quad (b)\ \ \mu_{r,d}\,\widetilde{\nabla}\psi_m = -L\,\mathbf{rot}\,\varphi_m, \quad \text{in } \widehat{D}, \tag{33}$$

where $\widetilde{\mathbf{rot}} = -e_3 \times \widetilde{\nabla}$. Note that from the relation above one can deduce another expression of the resistance coefficient R, namely (see (29) for the definition of $\beta_{r,c}$)

$$R = L^2 \int_{\partial\widehat{D}} \beta_c \left|\mu_{r,d}^{-1} \nabla\varphi_m \cdot n\right|^2 \mathrm{d}s.$$

From this expression, we can conclude to the strict positivity of the resistance coefficient (in the spirit of Theorem 2 for the positivity conductance coefficient).

Theorem 3 *The resistance coefficient R is strictly positive.*

Proof Assume by contradiction that $R = 0$. Hence $\mu_{r,d}^{-1}\nabla\varphi_m \cdot n = 0$ along Σ_+. Since φ_m also vanishes along Σ_+ the unique continuation principle applies again (see [22]) and one deduces $\varphi_m = 0$ in $\widehat{D}$. But this contradicts the boundary conditions in (30). ■

3.2 Telegrapher's equations

The goal of this paper is to derive, by asymptotic analysis, a reduced 1D model of the coaxial cable from Maxwell's equation (14) when the small parameter δ tends to zero. Such 1D model describes the propagation of the electrical voltage V_{ef}^{δ} and current I_{ef}^{δ} which are the solutions of the telegrapher's equations with losses

$$\begin{cases} C\,\partial_t V_{\text{ef}}^{\delta} + \delta\, G\, V_{\text{ef}}^{\delta} + \partial_3 I_{\text{ef}}^{\delta} = J^{\delta} \ \text{ in } \mathbb{R}\times(0,T), \\[2mm] L\,\partial_t I_{\text{ef}}^{\delta} + \delta\, R\, \partial_t^{\frac{1}{2}} I_{\text{ef}}^{\delta} + \partial_3 V_{\text{ef}}^{\delta} = 0 \text{ in } \mathbb{R}\times(0,T), \end{cases} \tag{34}$$

with null initial condition and where the coefficients RLCG are defined in (26, 27, 29), the source term is given by

$$J^{\delta} = \int_{\widehat{D}} \mathbf{j}_T^{\delta}\cdot\nabla\varphi_e \,\mathrm{d}x$$

and the half-time derivative is taken is the sense of Caputo [18], that is defined as

$$\forall f \in W^{1,\infty}(\mathbb{R}^+), \quad \partial_t^{\frac{1}{2}} f(t) = \frac{1}{\sqrt{\pi}}\int_0^t \frac{\partial_t f(\tau)}{(t-\tau)^{\frac{1}{2}}}\,\mathrm{d}\tau \in C^0([0,T]). \tag{35}$$

It is well known that the half-derivative terms in (34) is responsible for non-local phenomena in time that are dissipative and dispersive. Such phenomena was also observed in the effective telegrapher's equations that was obtained in [9]. However they are different in nature. On one hand, in [9] these phenomena resulted from strong conductivity – in $O(1)$ in δ – in the dielectric. On the other hand in this present work, the dissipative and dispersive properties come from the very strong conductivity – in $O(\delta^{-4})$ – in the conducting domain.

4 An asymptotic ansatz

4.1 Preliminary geometrical tools

In what follows and in the rest of the manuscript we drop the $\hat{\cdot}$ *notation to denote space variables.*

In this paragraph, we work in the plane $\mathbb{R}^2$, with current point $x=(x_1,x_2)$, seen as the orthogonal in $\mathbb{R}^3$ to the unit vector $\mathbf{e}_3 := (0,0,1)^t$. In particular, assimilating $x\in\mathbb{R}^2$ to $(x,0)^t\in\mathbb{R}^3$, the rotation of angle $\frac{\pi}{2}$ corresponds to $x\mapsto \mathbf{e}_3\times x$.

Parametrization of the interfaces. In the following, we shall assume that the two interfaces $\Sigma_\pm$ are two closed curves of class C^2 that can be parametrized by their curvilinear abscissa, namely,

$$\Sigma_\pm = \big\{x_\pm(s)\in\mathbb{R}^2, s\in[0,L^\pm),\ x_\pm(s)\in C^2_{per}(0,L^\pm;\mathbb{R}^2),\ |\dot{x}_\pm(s)|=1\big\}, \tag{36}$$

where $L^\pm = |\Sigma_\pm|$, $\dot{x}_\pm$ holds for the derivative of $x_\pm$ and

$$C^2_{per}(0,L^\pm;\mathbb{R}^2) := \big\{y\in C^2(\mathbb{R};\mathbb{R}^2)\ /\ y(s+L^\pm)=y(s)\big\}.$$

Moreover, we assume that

$$x_\pm(s) \text{ is bijective from } [0, L_\pm) \text{ into } \Sigma_\pm\ .$$

The unit tangent and normal vector along $\Sigma_\pm$ at point $x_\pm(s)$ are given by

$$\tau_\pm(s) = \dot{x}_\pm(s), \quad n_\pm(s) = \mathbf{e}_3 \times \tau_\pm(s),$$

and the algebraic curvature $c_\pm(s)$ of $\Sigma_\pm$ at point $x_\pm(s)$ is defined by

$$\dot{\tau}_\pm(s) = c_\pm(s)\, n_\pm(s) \quad \text{which gives } c_\pm = \ddot{x}_{2,\pm}\, \dot{x}_{1,\pm} - \ddot{x}_{1,\pm}\, \dot{x}_{2,\pm}\ .$$

All the above quantities depend on the orientation of $\Sigma_\pm$: by convention, the orientation of $\Sigma_\pm$ is chosen in such a way that the normal vector $n_\pm(s)$ points inside the domain $\widehat{C}^\pm$ and outside $\widehat{D}$, according to Figure 2.

Tubular subdomains and local coordinates. Let us introduce the maps (of class C^2)

$$F_\pm : \mathbb{R}^2 \to \mathbb{R}^2 \ \text{ such that } \ F_\pm(s,\nu) := x_\pm(s) + \nu\, n_\pm(s).$$

It is easy to check that the jacobian $J_\pm$ of $F_\pm$ is given by

$$J_\pm(s,\nu) = 1 - \nu\, c_\pm(s).$$

Thus the restriction of $F_\pm$ to the domain $\mathcal{O}_\pm := \{(s,\nu)\ /\ 1 - \nu\, c_\pm(s) > 0\}$ is locally injective. By construction, $\Sigma_\pm$ is the image by $F_\pm$ of $[0, L_\pm) \times \{0\}$ and, from the orientation of $\Sigma_\pm$ (previous paragraph) we know that for each $s \in [0, L^\pm)$, for ν small enough , $F_\pm(s,\nu) \in \widehat{C}^\pm$. From the above considerations there exists $0 < d_\pm < \inf c_\pm(s)$ such that introducing the tubular domain

$$\widehat{C}^\pm_{int} := \big\{F_\pm(s,\nu), s \in [0, L^\pm), 0 < \nu < d_\pm\big\} \tag{37}$$

(which we call a one-sided neirghborhood of $\Sigma_\pm$), then

$$F_\pm \text{ is a bijection from } \mathcal{R}_\pm := [0, L^\pm) \times (0, d_\pm) \text{ into } \widehat{C}^\pm_{int}. \tag{38}$$

In other words $(s,\nu) \in \mathcal{R}_\pm$ is a system of coordinates in the domains $\widehat{C}^\pm_{int}$.

For convenience, we also introduce the domains

$$\widehat{C}^\pm_{ext} := \widehat{C}^\pm \setminus \widehat{C}^\pm_{int}. \tag{39}$$

2D Differential operators in local coordinates. To a function $\varphi : \widehat{C}^\pm_{int} \mapsto \mathbb{R}^n$, we associate the function $\breve{\varphi} : \mathcal{R}_\pm \mapsto \mathbb{R}^n$ defined by

$$\forall\ (s,\nu) \in \mathcal{R}_\pm, \quad \breve{\varphi}(s,\nu) = \varphi\big(F_\pm(s,\nu)\big) = \big(\varphi \circ F_\pm\big)(s,\nu). \tag{40}$$

We know recall the action of $2D$-differential operators in local coordinates. We begin with the gradient and vector rotational of a scalar field $\varphi \in C^1\big(\widehat{C}^\pm_{int}\big)$,

$$\begin{cases} (\nabla\varphi) \circ F_\pm = J_\pm^{-1}\ (\partial_s\breve{\varphi})\, \tau + (\partial_\nu\breve{\varphi})\, n, \\[2ex] (\mathbf{rot}\, \varphi) \circ F_\pm = -J_\pm^{-1}\ (\partial_s\breve{\varphi})\, n + (\partial_\nu\breve{\varphi})\, \tau, \end{cases} \tag{41}$$

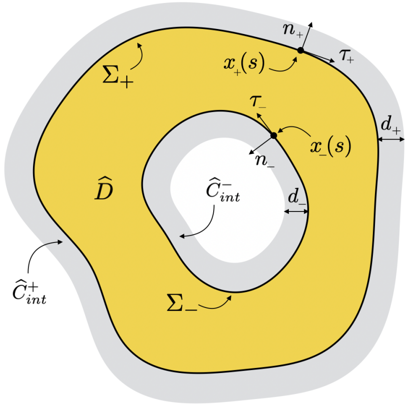

Fig. 2 Parametrization of the interfaces and local coordinates

and pursue with the divergence and rotational of a 2D vector field $u \in C^1\big(\widehat{C}^{\pm}_{int}\big)^2$,

$$\left\{\begin{array}{l}(\text{div } u) \circ F_\pm = J_\pm^{-1}\,(\partial_s \breve{u})\cdot\tau + (\partial_\nu \breve{u})\cdot n,\\[2ex](\text{rot } u)\circ F_\pm = J_\pm^{-1}\,(\partial_s \breve{u})\cdot n - (\partial_\nu \breve{u})\cdot\tau.\end{array}\right. \tag{42}$$

Maxwell's equations in the conductors in local coordinates. In the tubular domains $\widehat{C}^{\pm}_{int}\times\mathbb{R}$, we can rewrite the (rescaled) Maxwell's equations (14) using the local coordinates (s,ν). Below, we use obvious notations, consistent with (40),

$$\breve{E}^\delta_T(s,\nu,x_3,t) = E^\delta_T\big(F_\pm(s,\nu),x_3,t\big),\ \breve{E}^\delta_s(s,\nu,x_3,t) = E^\delta_3\big(F_\pm(s,\nu),x_3,t\big),\cdots$$

where the space variable (s,ν,x_3) varies in $\mathcal{Q}_\pm := \mathcal{R}_\pm\times\mathbb{R}$.

Then, taking into account (13), it is easily seen from (41, 42) that (14) becomes

$$\left\{\begin{array}{l}\varepsilon_{r,c}\,\partial_t\breve{E}^\delta_T + \dfrac{\sigma_{r,c}}{\delta^4}\,\breve{E}^\delta_T - \partial_3\,(\mathbf{e}_3\times\breve{H}^\delta_T) + \dfrac{J_\pm^{-1}}{\delta}\,(\partial_s\breve{H}^\delta_3)\,n - \dfrac{1}{\delta}\,(\partial_\nu\breve{H}^\delta_3)\,\tau = \mathbf{0},\\[2ex]\varepsilon_{r,c}\,\partial_t\breve{E}^\delta_3 + \dfrac{\sigma_{r,c}}{\delta^4}\,E^\delta_3 + \dfrac{1}{\delta}\,(\partial_\nu\breve{H}^\delta_T)\cdot\tau - \dfrac{J_\pm^{-1}}{\delta}\,(\partial_s\breve{H}^\delta_T)\cdot n = 0,\\[2ex]\mu_{r,c}\,\partial_t\breve{H}^\delta_T + \partial_3\,(\mathbf{e}_3\times\breve{E}^\delta_T) - \dfrac{J_\pm^{-1}}{\delta}\,(\partial_s\breve{E}^\delta_3)\,n + \dfrac{1}{\delta}\,(\partial_\nu\breve{E}^\delta_3)\,\tau = \mathbf{0},\\[2ex]\mu_{r,c}\,\partial_t\breve{H}^\delta_3 - \dfrac{1}{\delta}\,(\partial_\nu\breve{E}^\delta_T)\cdot\tau + \dfrac{J_\pm^{-1}}{\delta}\,(\partial_s\breve{E}^\delta_T)\cdot n = 0,\end{array}\right. \tag{43}$$

where (i) all the equations are set in $\mathcal{Q}_\pm$, (ii) the fields are implicitly $L_\pm$-periodic in s, (iii) τ and n are considered as vector fields in $\mathcal{Q}_\pm$ defined by

$$\tau(s,\nu,x_3)\equiv\tau_\pm(s),\quad n(s,\nu,x_3)\equiv n_\pm(s),\quad \forall\,(s,\nu,x_3)\in\mathcal{Q}_\pm.$$

4.2 A transmission problem in the frequency domain

We wish to analyze the asymptotic behaviour when δ tends to 0 of the solution of the rescaled problem (14). To do so, in the spirit of [9], we first pass to the frequency domain (which corresponds to consider the Fourier transforms of the unknown fields). For simplicity, we use the same letters for writing the fields in the time and frequency domains, assuming that the context will be sufficient to dissipate any ambiguity

$$E_T^\delta(x, x_3, t) \to E_T^\delta(x, x_3, \omega), \quad E_3^\delta(x, x_3, t) \to E_3^\delta(x, x_3, \omega), \quad \cdots$$

Next, for each frequency $\omega > 0$ which becomes a parameter, we formulate a transmission problem between

- the dielectric domain $\widehat{\Omega}_d = D \times \mathbb{R}$,
- the cylindrical tubular domains $\widehat{C}^{\pm}_{int} \times \mathbb{R}$.

Moreover, for simplicity, the dependence of all the unknowns with respect to the frequency ω will not be mentioned explicitly. This convention will be kept up to the beginning of section 6.

Implicitly, we shall not look at the solution in the *exterior* domains $\widehat{C}^{\pm}_{ext} \times \mathbb{R}$ (see (39)), i. e. inside the conductor but sufficiently far from the interfaces with the dielectric. This is justified by the skin effect: mathematically, it is possible to show that, for any $\omega > 0$, the fields are exponentially small in these domains when $\delta \to 0$ (see for instance [8], Theorem 2.2). The transmission problem reads:

$$\text{Find } \begin{cases} (E_T^\delta, E_3^\delta), (H_T^\delta, H_3^\delta) : \widehat{\Omega}_d \to \mathbb{C}^3 \times \mathbb{C}^3, \quad \Omega_d = D \times \mathbb{R}, \\[2ex] (\breve{E}_T^\delta, \breve{E}_3^\delta), (\breve{H}_T^\delta, \breve{H}_3^\delta) : \mathcal{Q}_\pm \to \mathbb{C}^3 \times \mathbb{C}^3, \;\; \mathcal{Q}_\pm = \mathcal{R}_\pm \times \mathbb{R}, \end{cases}$$

such that

$$\begin{cases} i\omega\, \varepsilon_{r,d}\, E_T^\delta + \delta\, \sigma_{r,d}\, E_T^\delta - \partial_3\, (\mathbf{e}_3 \times H_T^\delta) - \delta^{-1}\, \mathbf{rot}\; H_3^\delta = \mathbf{j}_T^\delta, & \text{in } \widehat{\Omega}_d, \;(44a) \\[2ex] i\omega\, \varepsilon_{r,d}\, E_3^\delta + \delta\, \sigma_{r,d}\, E_3^\delta - \delta^{-1}\, \mathrm{rot}\; H_T^\delta = j_3^\delta, & \text{in } \widehat{\Omega}_d, \;(44b) \\[2ex] i\omega\, \mu_{r,d}\, H_T^\delta + \partial_3\, (\mathbf{e}_3 \times E_T^\delta) + \delta^{-1}\, \mathbf{rot}\; E_3^\delta = \mathbf{0}, & \text{in } \widehat{\Omega}_d, \;(44c) \\[2ex] i\omega\, \mu_{r,d}\, H_3^\delta + \delta^{-1}\, \mathrm{rot}\; E_T^\delta = 0, & \text{in } \widehat{\Omega}_d, \;(44d) \end{cases}$$

and

$$
\begin{cases}
i\omega\,\varepsilon_{r,c}\,\breve{E}_T^\delta + \dfrac{\sigma_{r,c}}{\delta^4}\,\breve{E}_T^\delta - \partial_3\,(\mathbf{e}_3\times\breve{H}_T^\delta) \\[2mm]
\qquad\qquad + \dfrac{J_\pm^{-1}}{\delta}\,(\partial_s\breve{H}_3^\delta)\,n - \dfrac{1}{\delta}\,(\partial_\nu\breve{H}_3^\delta)\,\tau = \mathbf{0}, & \text{in } \mathcal{Q}_\pm, \\[2mm]
i\omega\,\varepsilon_{r,c}\,\breve{E}_3^\delta + \dfrac{\sigma_{r,c}}{\delta^4}\,\breve{E}_3^\delta + \dfrac{1}{\delta}\,(\partial_\nu\breve{H}_T^\delta)\cdot\tau - \dfrac{J_\pm^{-1}}{\delta}\,(\partial_s\breve{H}_T^\delta)\cdot n = 0, & \text{in } \mathcal{Q}_\pm, \\[2mm]
i\omega\,\mu_{r,c}\,\breve{H}_T^\delta + \partial_3\,(\mathbf{e}_3\times\breve{E}_T^\delta) - \dfrac{J_\pm^{-1}}{\delta}\,(\partial_s\breve{E}_3^\delta)\,n + \dfrac{1}{\delta}\,(\partial_\nu\breve{E}_3^\delta)\,\tau = \mathbf{0}, & \text{in } \mathcal{Q}_\pm, \\[2mm]
i\omega\,\mu_{r,c}\,\breve{H}_3^\delta - \dfrac{1}{\delta}\,(\partial_\nu\breve{E}_T^\delta)\cdot\tau + \dfrac{J_\pm^{-1}}{\delta}\,(\partial_s\breve{E}_T^\delta)\cdot n = 0, & \text{in } \mathcal{Q}_\pm,
\end{cases}
\tag{45}
$$

coupled with the transmission conditions (understood in the sense of traces)

$$
\begin{cases}
E_T^\delta\big(x_\pm(s),x_3\big)\cdot\tau(s) = \breve{E}_T^\delta(s,0,x_3)\cdot\tau(s), & s\in[0,L_\pm), \\[2mm]
H_T^\delta\big(x_\pm(s),x_3\big)\cdot\tau(s) = \breve{H}_T^\delta(s,0,x_3)\cdot\tau(s), & s\in[0,L_\pm), \\[2mm]
E_3^\delta\big(x_\pm(s),x_3\big) = \breve{E}_3^\delta(s,0,x_3), & s\in[0,L_\pm), \\[2mm]
H_3^\delta\big(x_\pm(s),x_3\big) = \breve{H}_3^\delta(s,0,x_3), & s\in[0,L_\pm).
\end{cases}
\tag{46}
$$

for all $x_3\in\mathbb{R}$. Throughout the rest of this section, it will be not recalled explicitly that $x_3\in\mathbb{R}$. Mathematically speaking, the problem (44, 45, 46) is not properly posed since boundary conditions are missing for (45) at $\nu = d_\pm$. However, in the context of our asymptotic analysis, this missing condition will be replaced by the particular ansatz we shall choose for the asymptotic expansion of the fields (cf. Section 4.4 and more precisely (58)).

In addition, we shall use some hidden information consequence of (44, 45, 46). The first one is the divergence free property of the electromagnetic fields inside the dielectric, that follows directly from equation (2) after rescaling (here the property that the source term is divergence free in the rescaled coordinates (15, 16) is used implictely).

$$
\begin{cases}
\partial_3\big((i\omega\varepsilon_{r,d}+\delta\,\sigma_{r,d})E_3^\delta\big) + \dfrac{1}{\delta}\,\mathrm{div}\,\big((i\omega\varepsilon_{r,d}+\delta\,\sigma_{r,d})E_T^\delta\big) = 0 & \text{in } \widehat{\Omega}_d, \\[2mm]
\partial_3(\mu_{r,d}H_3^\delta) + \dfrac{1}{\delta}\,\mathrm{div}\,(\mu_{r,d}H_T^\delta) = 0 & \text{in } \widehat{\Omega}_d.
\end{cases}
\tag{47}
$$

We shall also use hidden transmission condition that are consequence, again, of equation (2). More precisely, since the fields $\varepsilon\,\partial_t\mathbf{E} + \sigma\,\mathbf{E}$ and $\mu\,\mathbf{H}$ are divergence free in whole domain, their normal traces should not jump. Hence introducing, for the sake of conciseness in the notation, the complex and frequency dependent permittivities $\varepsilon^{*,\delta}_{r,c}(\omega)$ and $\varepsilon^{*,\delta}_{r,d}(\omega)$ defined by

$$i\omega\, \varepsilon^{*,\delta}_{r,c}(\omega) := i\omega\, \varepsilon_{r,c} + \delta^{-4}\, \sigma_{r,d}, \quad i\omega\, \varepsilon^{*,\delta}_{r,d}(\omega) := i\omega\, \varepsilon_{r,d} + \delta\, \sigma_{r,d}, \tag{48}$$

we have the following transmission conditions in the local coordinates,

$$\begin{cases} \big[\, \varepsilon^{*,\delta}_{r,d}(\omega)\, E^\delta_T\big(x_\pm(s), x_3\big) - \varepsilon^{*,\delta}_{r,d}(\omega)\, \breve{E}^\delta_T(s,0,x_3)\,\big] \cdot n_\pm(s) = 0, & s \in [0, L_\pm), \\ \big[\, \mu_{r,d}\, H^\delta_T\big(x_\pm(s), x_3\big) - \mu_{r,c}\, \breve{H}^\delta_T(s,0,x_3)\,\big] \cdot n_\pm(s) = 0, & s \in [0, L_\pm). \end{cases} \tag{49}$$

4.3 A boundary layer rescaling of the equations in the conductors

Anticipating the presence of a boundary layer in the conductors, we apply a new change of variable in the domains changing ν into $\eta = \nu/\delta$. Doing so, we transform the fields $\big(\breve{E}(s,\nu), \breve{H}(s,\nu)\big)$ into $\big(\widetilde{E}^\delta(s,\eta), \widetilde{H}^\delta(s,\eta)\big)$ via

$$\breve{E}(s,\nu) = \widetilde{E}(s,\nu/\delta), \quad \breve{H}(s,\nu) = \widetilde{H}(s,\nu/\delta), \tag{50}$$

in such a way that the fields $(\widetilde{E}^\delta(s,\eta), \widetilde{H}^\delta(s,\eta))$ are defined in the domains $\mathcal{Q}^\delta_\pm := \mathcal{R}^\delta_\pm \times \mathbb{R}$ where $\mathcal{R}^\delta_\pm$ is the rectangle

$$\mathcal{R}^\delta_\pm := [0, L^\pm) \times (0, d_\pm/\delta).$$

Note that the domains $\mathcal{Q}^\delta_\pm$ increase when δ decreases and tend, when $\delta \to 0$ to the domain

$$\mathcal{Q}^0_\pm := [0, L^\pm) \times (0, +\infty) \times \mathbb{R}.$$

From (45), we deduce (using $\partial_\nu \longrightarrow \delta^{-1}\, \partial_\eta$) that in $\mathcal{Q}^\delta_\pm$, the fields $\big(\widetilde{E}^\delta_T, \widetilde{E}^\delta_3\big)$ and $\big(\widetilde{H}^\delta_T, \widetilde{H}^\delta_3\big)$ are solutions of the following system of equations,

$$\begin{cases} i\omega\varepsilon_{r,c}\, \widetilde{E}^\delta_T + \dfrac{\sigma_{r,c}}{\delta^4}\, \widetilde{E}^\delta_T - \partial_3(\mathbf{e}_3 \times \widetilde{H}^\delta_T) + \dfrac{J^{-1}_{\pm,\delta}}{\delta}\, (\partial_s \widetilde{H}^\delta_3)\, n - \dfrac{1}{\delta^2}\, (\partial_\eta \widetilde{H}^\delta_3)\, \tau = \mathbf{0}, & \text{(51a)} \\[2ex] i\omega\varepsilon_{r,c}\, \widetilde{E}^\delta_3 + \dfrac{\sigma_{r,c}}{\delta^4}\, \widetilde{E}^\delta_3 + \dfrac{1}{\delta^2}\, (\partial_\eta \widetilde{H}^\delta_T) \cdot \tau - \dfrac{J^{-1}_{\pm,\delta}}{\delta}\, (\partial_s \widetilde{H}^\delta_T) \cdot n = 0, & \text{(51b)} \\[2ex] i\omega\mu_{r,c}\, \widetilde{H}^\delta_T + \partial_3(\mathbf{e}_3 \times \widetilde{E}^\delta_T) - \dfrac{J^{-1}_{\pm,\delta}}{\delta}\, (\partial_s \widetilde{E}^\delta_3)\, n + \dfrac{1}{\delta^2}\, (\partial_\eta \widetilde{E}^\delta_3)\, \tau = \mathbf{0}, & \text{(51c)} \\[2ex] i\omega\mu_{r,c}\, \widetilde{H}^\delta_3 - \dfrac{1}{\delta^2}\, (\partial_\eta \widetilde{E}^\delta_T) \cdot \tau + \dfrac{J^{-1}_{\pm,\delta}}{\delta}\, (\partial_s \widetilde{E}^\delta_T) \cdot n = 0. & \text{(51d)} \end{cases}$$

where $J_{\pm,\delta}(s,\eta) := 1 - \delta\, \eta\, c_\pm(s)$. In the context of our asymptotic analysis, we shall simply use the fact that

$$J^{-1}_{\pm,\delta} = 1 + O(\delta). \tag{52}$$

For completeness, we also rewrite the equations(46) with the help of the new fields $(\widetilde{E}^\delta, \widetilde{H}^\delta)$, even though their writing is not affected by the boundary layer scaling: they are directly obtained from (46) by changing $(\breve{E}^\delta, \breve{H}^\delta)$ into $(\widetilde{E}^\delta, \widetilde{H}^\delta)$

$$\left\{\begin{array}{lll} E_T^\delta\big(x_\pm(s), x_3\big)\cdot\tau(s) = \widetilde{E}_T^\delta(s,0,x_3)\cdot\tau(s), & s\in[0,L_\pm), & \text{(53a)}\\[2ex] H_T^\delta\big(x_\pm(s), x_3\big)\cdot\tau(s) = \widetilde{H}_T^\delta(s,0,x_3)\cdot\tau(s), & s\in[0,L_\pm), & \text{(53b)}\\[2ex] E_3^\delta\big(x_\pm(s), x_3\big) = \widetilde{E}_3^\delta(s,0,x_3), & s\in[0,L_\pm), & \text{(53c)}\\[2ex] H_3^\delta\big(x_\pm(s), x_3\big) = \widetilde{H}_3^\delta(s,0,x_3), & s\in[0,L_\pm). & \text{(53d)} \end{array}\right.$$

Finally, to summarize, the transmission problem that we wish to analyze from the asymptotic point of view is the following

$$\text{Find}\ \left\{\begin{array}{l} (E_T^\delta, E_3^\delta), (H_T^\delta, H_3^\delta) : \widehat{\Omega}_d \to \mathbb{C}^3\times\mathbb{C}^3, \quad \widehat{\Omega}_d = \widehat{D}\times\mathbb{R},\\[2ex] (\widetilde{E}_T^\delta, \widetilde{E}_3^\delta), (\widetilde{H}_T^\delta, \widetilde{H}_3^\delta) : \mathcal{Q}_\pm^\delta \to \mathbb{C}^3\times\mathbb{C}^3, \ \ \mathcal{Q}_\pm^\delta = \mathcal{R}_\pm^\delta\times\mathbb{R}, \end{array}\right.$$

that satisfy (44) in $\widehat{\Omega}_d$, (51) in $\mathcal{Q}_\pm^\delta$ and the transmission conditions (53).

Finally observe that, from the transmission conditions (49) one can deduce, again by changing $(\breve{E}^\delta, \breve{H}^\delta)$ into $(\widetilde{E}^\delta, \widetilde{H}^\delta)$, the following transmissions conditions,

$$\left\{\begin{array}{lll} \big[\ \varepsilon_{r,d}^{*,\delta}(\omega)\, E_T^\delta\big(x_\pm(s), x_3\big) - \varepsilon_{r,d}^{*,\delta}(\omega)\, \widetilde{E}_T^\delta(s,0,x_3)\ \big]\cdot n_\pm(s) = 0, & s\in[0,L_\pm), & \text{(54a)}\\[2ex] \big[\ \mu_{r,d}\, H_T^\delta\big(x_\pm(s), x_3\big) - \mu_{r,c}\, \widetilde{H}_T^\delta(s,0,x_3)\ \big]\cdot n_\pm(s) = 0, & s\in[0,L_\pm). & \text{(54b)} \end{array}\right.$$

4.4 Asymptotic ansatz and orientation

In the dielectric domain, we shall look for a standard power series expansions of the unknown fields. This means that we look for

$$(E_T^k, E_3^k), (H_T^k, H_3^k) : \widehat{\Omega}_d \to \mathbb{C}^3\times\mathbb{C}^3, \quad k\in\mathbb{N}, \tag{55}$$

such that, at least formally,

$$(E_T^\delta, E_3^\delta) = \sum_{k=0}^{+\infty} \delta^k\,(E_T^k, E_3^k), \quad (H_T^\delta, H_3^\delta) = \sum_{k=0}^{+\infty} \delta^k\,(H_T^k, H_3^k) \quad \text{in } \widehat{\Omega}_d. \tag{56}$$

Similarly, in the domain $\mathcal{Q}_\pm^0$, we shall look for

$$(\widetilde{E}_T^k, \widetilde{E}_3^k), (\widetilde{H}_T^k, \widetilde{H}_3^k) : \mathcal{Q}_\pm^0 \to \mathbb{C}^3\times\mathbb{C}^3, \quad k\in\mathbb{N}, \tag{57}$$

where each field is assumed to tend to 0 for large η (this is where the expected skin depth is encoded)

$$\lim_{\eta\to+\infty} (\widetilde{E}_T^k, \widetilde{E}_3^k)(s,\eta,x_3) = 0, \qquad \lim_{\eta\to+\infty} (\widetilde{H}_T^k, \widetilde{H}_3^k)(s,\eta,x_3) = 0, \tag{58}$$

and such that, at least formally,

$$(\widetilde{E}_T^\delta, \widetilde{E}_3^\delta) = \sum_{k=0}^{+\infty} \delta^k \, (\widetilde{E}_T^k, \widetilde{E}_3^k), \quad (\widetilde{H}_T^\delta, \widetilde{H}_3^\delta) = \sum_{k=0}^{+\infty} \delta^k \, (\widetilde{H}_T^k, \widetilde{H}_3^k). \tag{59}$$

In both expansions (56) and (59) the terms of order 0 are the leading order fields when δ tends to 0, while the terms of order k for $k \geq 1$ are called the correctors. To determine a cable model accounting for dissipation effects, we will use not only the fields of order 0 but also the first order correctors in the dielectric as well as the second order correctors in the conductors. Indeed we shall see that, in the conductors, the limit electric field and the corresponding first order correctors will be identically 0. On the other hand, in the dielectric, the limit fields will be the same as in the case of perfect conductors (as for the model developed in [9]). Then we shall see that the correctors of order 1 are the first ones which are affected by the presence of the conductivity in both the dielectric and the conductors. In other words, in our model, the dissipation will appear as a first order effect (in δ) which is coherent with the fact that we want to model small losses. It is remarkable that the very strong conductivity in the conducting part will result for the electromagnetic field in the dielectric into a weak dissipation, of the same order of the ones due to the weak conductivity in the dielectric.

5 Characterization of the first terms of the asymptotic expansions

In order to characterise the first term in the expansions (56) and (59) we plug these expansions into (44) and (51) respectively and identify the terms in the same power δ. Ranking these powers by increasing values, we obtain a cascade of equations of orders $k = 0, 1, 2, \ldots$ both in the dielectric and the conductors. To close our problems we also use the transmission conditions (53) and (54) that will relate the two expansions of the solutions. Then the (standard) approach we adopt is the following: we look at the equations at order k by increasing values of k alternatively in the conductor (first) and in the dielectric. More precisely

- We first show that some terms of the leading electric and magnetic field expansion vanish in the conducting regions. More precisely

$$\widetilde{E}_T^0 = \widetilde{E}_T^1 = \mathbf{0}, \quad \widetilde{E}_3^0 = \widetilde{E}_3^1 = 0 \quad \text{and} \quad \widetilde{H}_T^0 \cdot n = 0 \quad \text{(Section 5.1)}.$$

- We deduce the specific structure of that the limit electromagnetic fields in the dielectric (in particular they are transversely polarised),

$$E_T^0 = V_0 \nabla \varphi_e, \quad H_T^0 = I_0 \widetilde{\nabla} \psi_m \quad \text{and} \quad E_3^0 = H_3^0 = 0,$$

  where V_0 and I_0 are the limit electric voltage and electric current and only depend on x_3. We show that they satisfy the so-called (lossless) telegrapher's equations in the frequency domain (Section 5.2).

- We show that the limit order longitudinal magnetic field and the tangential component of the second order electric field vanish in the conducting regions,

$$\widetilde{H}_3^0 = 0 \quad \text{and} \quad \widetilde{E}_T^2 = \mathbf{0} \quad \text{(Section 5.3)}.$$

– We then obtain an expression of the first order corrector of the longitudinal component of the electromagnetic fields (E_3^1, H_3^1) in the dielectric. An expression of these fields is given using only the harmonic potentials defined in Section 3.1, the limit voltage and current (V_0, I_0) and $\mathbf{j}_T^0$. (Section 5.4)

– Then we deduce that the fields $\widetilde{E}_3^2$, $\widetilde{H}_0^T \cdot \tau$ and $\widetilde{H}_1^T \cdot n$ decay exponentially in the conducting regions: their expression is completely determined by their value at the boundary of the dielectric. Thanks to transmission conditions, we are able to relate these values to the limit field $H_T^0 \cdot \tau$ (Section 5.5).

– Finally, we use the these equations to determine the first order electromagnetic correctors, E_T^1 and H_T^1 in the dielectric. By doing so, we introduce another telegrapher's equations, for the first order voltage V_1 and current I_1, that includes phenomena related to the dissipation in the dielectric and in the conducting domain (i.e. the skin-effect)(Section 5.6).

5.1 Leading order terms in the conducting region : $\widetilde{E}^0$, $\widetilde{E}^1$ and $\widetilde{H}^0 \cdot n$

Due to the presence of the δ^{-4} factor in the equations (51), the first directly available information concerns the electric field in the conducting region. More precisely, after substituting (59) into (51a) and (51b) (namely the equations for the electric fields), and identifying term in δ^{-4} and δ^{-3}, we obtain directly for the electric field that

$$\widetilde{E}_T^0 = \widetilde{E}_T^1 = \mathbf{0} \quad \text{and} \quad \widetilde{E}_3^0 = \widetilde{E}_3^1 = 0, \quad \text{in } \mathcal{Q}_\pm^0. \tag{60}$$

which implies that the electric field will be of order $O(\delta^2)$ in the conducting region. This was expected since the electric field is expected to be very small in strongly conducting domains.

The leading order term for the electric field will thus be $\widetilde{E}^2$. Let us draw the reader's attention that the δ^{-2} terms do not involve the electric field only, but also the magnetic field, reason why the examination of this term is postponed to a next step of the identification process.

Equations (51c) and (51d), namely the equations for the magnetic field, does not provide a similar direct information on the magnetic field. We see that in these equations, as a consequence of (60) which automatically cancels the δ^{-2} and δ^{-1} terms, the leading order terms become the $O(1)$ term. For instance, the $O(1)$ term of (51c) gives, taking (60) into account,

$$i\omega\, \mu_{r,c}\, \widetilde{H}_T^0 + (\partial_\eta \widetilde{E}_3^2)\, \tau = 0. \tag{61}$$

To eliminate $\widetilde{E}_3^2$, which is not known at this stage of the process, we can simply take the scalar product of (61) by n, which shows that the normal component of the limit transverse magnetic field vanishes

$$\widetilde{H}_T^0 \cdot n = 0, \quad \text{in } \mathcal{Q}_\pm^0. \tag{62}$$

However, no information is obtained for the tangential field $\widetilde{H}_T^0 \cdot \tau$. In a similar manner, the $O(1)$ term of (51d) gives

$$i\omega\, \mu_{r,c}\, \widetilde{H}_3^0 + (\partial_\eta \widetilde{E}_T^2) \cdot \tau = 0, \tag{63}$$

which does not provide any direct information for $\widetilde{H}_3^0$.

5.2 The limit electromagnetic field in the dielectric : E^0 and H^0

The longitudinal fields. Substituting (56) into (44) and identifying the (leading order) δ^{-1} terms in in the first and third equations of (44) (the ones for the longitudinal fields), we obtain

$$\mathbf{rot}\, H_3^0 = \mathbf{0} \quad \text{and} \quad \mathbf{rot}\, E_3^0 = \mathbf{0}, \quad \text{in } \widehat{\Omega}_d, \tag{64}$$

which mean that E_3^0 and H_3^0 are constant in each cross section (x_3 constant) of $\widehat{\Omega}_d$. To determine these constants, the idea is to exploit the informations (in the conducting regions) of Section 5.1 through the transmission conditions (53). More precisely, the $O(1)$ terms in (53c) and (53d) (transmission conditions for the electric field) give, using (60),

$$(a) \quad E_3^0 = 0, \quad (b) \quad E_T^1 \cdot \tau = 0 \quad \text{on } \partial\widehat{\Omega}_d \tag{65}$$

In other words, the limit electric field satisfies the perfectly conducting boundary conditions and we can proceed as in [9] Since E_3^0 is constant, (65)(a) yields $E_3^0 = 0$ in $\widehat{\Omega}_d$. For H_3^0, the approach is more indirect. We write the $O(1)$ term of equation (44d), i.e. the equation for H_3^δ, which makes appear explicitly H_3^0,

$$i\omega\, \mu_r\, H_3^0 + \text{rot}\, E_T^1 = 0. \tag{66}$$

Finally, to exploit (65)(b), we simply integrate the above equation over the cross section $\widehat{D}$ (for each x_3), using (18) we get,

$$i\omega \int_{\widehat{D}} \mu_r\, H_3^0\, \mathrm{d}x = \int_{\partial\widehat{D}} E_T^1 \times n\, \mathrm{d}s = 0 \quad (\text{by (65)(b)}). \tag{67}$$

Again, since H_3^0 is constant in each cross section, we have proven that $H_3^0 = 0$ in $\widehat{\Omega}_d$. We have finally obtained that the limit electromagnetic field in the dielectric region is transversely polarized, namely

$$H_3^0 = 0 \quad \text{and} \quad E_3^0 = 0, \tag{68}$$

as it is usually assumed in the engineering literature (see [14]).

Structure of the transverse fields. Substituting (56) into equations (44b) and (44d), the ones for the transverse fields, and identifying the (leading order) δ^{-1} terms, we obtain

$$\mathrm{rot}\; H_T^0 = 0 \quad \text{and} \quad \mathrm{rot}\; E_T^0 = 0, \quad \text{in } \widehat{\Omega}_d. \tag{69}$$

Proceeding similarly in the hidden divergence equation (47) we obtain

$$\mathrm{div}\; \varepsilon_{r,d}\, E_T^0 = 0 \quad \text{and} \quad \mathrm{div}\; \mu_{r,d}\, H_T^0 = 0, \quad \text{in } \widehat{\Omega}_d. \tag{70}$$

Finally, substituting (56) and (59) into (53) and the second equation of (54) and identifying term in $O(1)$, we obtain using (60, 62) the following equalities that hold at the boundary of the dielectric,

$$E_T^0 \cdot \tau = 0, \quad \text{and} \quad \mu_{r,d}\, H_T^0 \cdot n = 0, \tag{71}$$

Equations (69, 70, 71) are sufficient to characterise the structure of the limit fields. More precisely, according to Theorem 1 that, there exists functions

$$V_0 : \mathbb{R} \to \mathbb{C} \qquad \text{and} \quad I_0 : \mathbb{R} \to \mathbb{C},$$

representing the electric voltage and electric current respectively, such that

$$E_T^0(x, x_3) = V_0(x_3) \nabla \varphi_e(x), \quad H_T^0(x, x_3) = I_0(x_3) \widetilde{\nabla} \psi_m(x). \tag{72}$$

Equations for the electric voltage and current. Substituting (56) into (44) and identifying terms in $O(1)$ in (44a) and (44c) we obtain

$$\begin{cases} i\omega\, \varepsilon_r\, E_T^0 - \partial_3\, (\mathbf{e}_3 \times H_T^0) - \mathbf{rot}\; H_3^1 = \mathbf{j}_T^0, \\ i\omega\, \mu_r\, H_T^0 + \partial_3\, (\mathbf{e}_3 \times E_T^0) + \mathbf{rot}\; E_3^1 = \mathbf{0}. \end{cases} \tag{73}$$

This does not provide closed equations for (E_T^0, H_T^0) due to the presence of (E_3^1, H_3^1). To get rid of them, as in [9] we take the scalar product in $L^2(\widehat{D})$ of the two equations of (73) (written for any x_3) with $\nabla \varphi_e$ and $\widetilde{\nabla} \psi_m$ respectively. Using the expression (72) of E_T^0 and H_T^0, the property (25) and the definition (26) of the coefficients C and L, we then get

$$\begin{cases} i\omega\, C\, V_0 + \partial_3 I_0 - \displaystyle\int_{\widehat{D}} \nabla \varphi_e \cdot \mathbf{rot}\; H_3^1 \mathrm{d}x = \int_{\widehat{D}} \nabla \varphi_e \cdot \mathbf{j}_T^0\, \mathrm{d}x, \\ i\omega\, L\, I_0 + \partial_3 V_0 + \displaystyle\int_{\widehat{D}} \widetilde{\nabla} \psi_m \cdot \mathbf{rot}\; E_3^1\, \mathrm{d}x = \mathbf{0}, \end{cases} \tag{74}$$

Finally, we remark that the terms involving E_3^1 and H_3^1 in the above equations vanish. Indeed, using the Green's formula (18),

$$\int_{\widehat{D}} \nabla \varphi_e \cdot \mathbf{rot}\; H_3^1\, \mathrm{d}x = \int_{\partial \widehat{D}} \nabla \varphi_e \times n \cdot H_3^1\, \mathrm{d}\sigma = 0,$$

since φ_e is constant on $\partial \widehat{D}$. Moreover, the transmission conditions (46) show that $E_3^1 = \widetilde{E}_3^1 = 0$ on $\partial \widehat{D}$, (because of (60)). Therefore, using (18) again,

$$\int_{\widehat{D}} \widetilde{\nabla} \psi_m \cdot \mathbf{rot}\; E_3^1\, \mathrm{d}x = \int_{\partial \widehat{D}} \widetilde{\nabla} \psi_m \times n \cdot E_3^1\, \mathrm{d}x = 0.$$

To sum-up, using (25), one can show that Equation (74) reads

$$\begin{cases} i\omega\, C\, V_0 + \partial_3 I_0 = J_0, \\ i\omega\, L\, I_0 + \partial_3 V_0 = 0, \end{cases} \tag{75}$$

where the scalar source current J_0 is given by

$$J_0 = \int_{\widehat{D}} \nabla\varphi_e \cdot \mathbf{j}_T^0 \,\mathrm{d}x. \tag{76}$$

5.3 Coming back to the conductors

We now come back to the expansion in the conductors. In section 5.1, we got that $\widetilde{E}^0 = \widetilde{E}^1 = 0$ as well as $\widetilde{H}_T^0 \cdot n = 0$. We are now going to see that $\widetilde{H}_3^0$ and $\widetilde{E}_T^2$ also vanish. In section 5.1, we already obtained an equation relating these two fields from the $O(1)$ term of (51d), namely

$$i\omega\, \mu_{r,c}\, \widetilde{H}_3^0 = (\partial_\eta \widetilde{E}_T^2) \cdot \tau = 0, \tag{77}$$

It is then easy to see that a second relationship between these two fields is obtained from the δ^{-2} term of equation (51a) which gives

$$\sigma_{r,c}\, \widetilde{E}_T^2 = \left(\partial_\eta \widetilde{H}_3^0\right) \tau. \tag{78}$$

Substituting (78) into (77), we obtain that $\widetilde{H}_3^0$ satisfies

$$i\omega\, \mu_{r,c}\, \sigma_{r,c}\, \widetilde{H}_3^0 - \partial_\eta^2 \widetilde{H}_3^0 = 0 \quad \text{in } \mathcal{Q}_\pm^0. \tag{79}$$

Because of the transmission condition (53d) and the equation (68) we have $\widetilde{H}_3^0$ at $\eta = 0$ hence $\widetilde{H}_3^0$ identically vanishes and so does $\widetilde{E}_T^2 \cdot \tau$. We have

$$\widetilde{H}_3^0 = 0 \quad \text{and} \quad \widetilde{E}_T^2 \cdot \tau = 0. \tag{80}$$

It suffices then to know the value of $\widetilde{H}_3^0$ for $\eta = 0$, which is provided by the transmission condition (53d)

$$\widetilde{H}_3^0(s, x_3, 0) = H_3^0(x_\pm(s), x_3) = 0 \quad (\text{since } H_3^0 \text{ vanishes in } \widehat{\Omega}_d, \text{ cf. } (60)) \tag{81}$$

From (79) and (81), since no exponential increase is allowed inside the conductor, we deduce that $\widetilde{H}_3^0$ vanishes identically in $\mathcal{Q}_\pm$, thus $\widetilde{E}_T^2$ too by (78):

$$\widetilde{H}_3^0 = 0, \quad \widetilde{E}_T^2 = \mathbf{0} \quad \text{in } \mathcal{Q}_\pm. \tag{82}$$

5.4 The first order longitudinal electromagnetic field in the dielectric

We now come back to the longitudinal fields whose leading orders become, because of (68), E_3^1 and H_3^1. We first notice that using (33), equation (73) can be recast in a form that only involve rotationals instead of gradients, namely

$$\left\{\begin{array}{ll}\mathbf{rot}\, H_3^1 = i\omega\, C\, V_0\, \widetilde{\mathbf{rot}}\, \psi_e + (\partial_3 I_0)\, \widetilde{\mathbf{rot}}\, \psi_m + I_0\, \widetilde{\mathbf{rot}}\, (\partial_3 \psi_m) - \mathbf{j}_T^0, & (a)\\ \mathbf{rot}\, E_3^1 = i\omega\, L\, I_0\, \mathbf{rot}\, \varphi_m + (\partial_3 V_0)\, \mathbf{rot}\, \varphi_e + V_0\, \mathbf{rot}\, (\partial_3 \varphi_e). & (b)\end{array}\right. \tag{83}$$

Next, we use (75) to replace $i\omega\, C\, V_0$ (resp. $i\omega\, L\, I_0$) in function of $\partial_3 I_0$ (resp. $\partial_3 V_0$) in (83)(a) (resp. (83)(b)). These manipulations allow us to express $\mathbf{rot}\, H_3^1$ (resp. $\mathbf{rot}\, E_3^1$) only in terms of I_0 (resp. V_0),

$$\left\{\begin{array}{ll}\mathbf{rot}\, H_3^1 = \partial_3 I_0\, \widetilde{\mathbf{rot}}\, (\psi_m - \psi_e) + I_0\, \widetilde{\mathbf{rot}}\, (\partial_3 \psi_m) - \mathbf{j}_T^e, & (a)\\ \mathbf{rot}\, E_3^1 = \partial_3 V_0\, \mathbf{rot}\, (\varphi_e - \varphi_m) + V_0\, \mathbf{rot}\, (\partial_3 \varphi_e), & (b)\end{array}\right. \tag{84}$$

where we have set

$$\mathbf{j}_T^e := \mathbf{j}_T^0 - J_0\, \widetilde{\mathbf{rot}}\, \psi_e. \tag{85}$$

For E_3^1, using (84)(b) and the connexity of $\widehat{D}$, we deduce the existence of a constant (in x) $C_e = C_e(x_3)$ such that

$$E_3^1 = \partial_3 V_0\; (\varphi_e - \varphi_m) + V_0\; \partial_3 \varphi_e + C_e. \tag{86}$$

Concerning H_3^1, for dealing in a similar way with (84)(a) we would like to rewrite $\mathbf{j}_T^e$ as a rotational. This requires preliminary remarks. First we observe that, since $\psi_m - \psi_e$ and $\partial_3 \psi_m$ do not jump across Γ,

$$\widetilde{\mathbf{rot}}\, (\psi_m - \psi_e) = \mathbf{rot}\; (\psi_m - \psi_e), \quad \widetilde{\mathbf{rot}}\, (\partial_3 \psi_m) = \mathbf{rot}\; (\partial_3 \psi_m). \tag{87}$$

Moreover, using (33) again, we see that (85) gives

$$\mathbf{j}_T^e = \mathbf{j}_T^0 - C^{-1} J_0\; \varepsilon_{r,d}\, \nabla \varphi_e, \quad J_0 = \int_{\widehat{D}} \nabla \varphi_e \cdot \mathbf{j}_T^0\, \mathrm{d}x \quad \text{(cf. (76))} \tag{88}$$

Referring to the definition (26) of C, we see that

$$\int_{\widehat{D}} \mathbf{j}_T^e \cdot \nabla \varphi_e\, \mathrm{d}x = 0. \tag{89}$$

Since $\mathbf{j}_T^0$ (by assumption) and $\varepsilon_{r,d}\, \nabla \varphi_e$ (by (19)) are divergence free, so is $\mathbf{j}_T^e$, i.e.,

$$\mathrm{div}\, \mathbf{j}_T^e = 0. \tag{90}$$

Finally, the flux of $\mathbf{j}_T^e$ across each connected component of $\partial \widehat{D}$ vanishes, i.e.

$$\int_{\Sigma_-} \mathbf{j}_T^e \cdot n = \int_{\Sigma_+} \mathbf{j}_T^e \cdot n = 0. \tag{91}$$

The proof of (91) is as follows. First integrating (90) in $\widehat{D}$ gives

$$\int_{\Sigma_-} \mathbf{j}_T^e \cdot n = - \int_{\Sigma_+} \mathbf{j}_T^e \cdot n$$

Next multiplying (90) by φ_e ant integrating the result over $\widehat{D}$ gives

$$-\int \mathbf{j}_T^e \cdot \nabla\varphi_e \, \mathrm{d}x + \int_{\partial\widehat{D}} (\mathbf{j}_T^e.n)\, \varphi_e = 0$$

Using (89), $\varphi_e = 0$ on Σ_- and $\varphi_e = 1$ on Σ_+, we simply get

$$\int_{\Sigma_+} \mathbf{j}_T^e \cdot n = 0$$

and (91) follows easily. Thus, thanks to (90, 91) and the connexity of $\widehat{D}$ and can use Theorem 3.1 of [13] to state that there exists a unique scalar field $\mathrm{H}_3^s \in H^1(\widehat{D})$ such that

$$\text{(i)} \quad \mathbf{j}_T^e = \mathbf{rot}\, \mathrm{H}_3^s, \quad \text{(ii)} \quad \int_{\widehat{D}} \mu_r \, \mathrm{H}_3^s \, \mathrm{d}x = 0. \tag{92}$$

From (92), (87) and (84)(a), we then deduce the existence of a constant (in x) $C_m = C_m(x_3)$ such that

$$H_3^1 = \mathrm{H}_3^s + \partial_3 I_0 \, (\psi_m - \psi_e) + I_0 \, \partial_3\psi_m + C_m. \tag{93}$$

To entirely determine E_3^1 and H_3^1, it remains to compute the constants C_e and C_m appearing in (86) and (93). They will be obtained by looking at what happens along the boundary $\partial\widehat{D}$ and this is where we are going to use the transmission conditions with the conductors.

First, since $\widetilde{E}_3^1 = 0$ in $\mathcal{Q}_\pm^0$, we know that E_3^1 vanishes on $\partial\widehat{D}$.

Since $(\varphi_m - \varphi_e)$ and $\partial_3\psi_m$ both vanish along $\partial\widehat{D}$ (cf. (19) and (30)), formula (86) gives $E_3^1 = C_e$ on $\partial\widehat{D}$. We thus conclude that $C_e = 0$.

The determination of C_m is slightly more involved. We proceed as in section 5.2 for H_3^0. The δ-term of (44d) gives

$$i\omega \, \mu_r \, \widehat{H}_3^1 + \mathrm{rot}\; E_T^2 = 0. \tag{94}$$

If we integrate this equation in $\widehat{D}$, we obtain, using (18),

$$i\omega \int_{\widehat{D}} \mu_r \, \widehat{H}_3^1 \, \mathrm{d}x = \int_{\partial\widehat{D}} E_T^2 \times n \, \mathrm{d}s = 0. \tag{95}$$

However, from the transmission condition (53a), $E_T^2 \cdot \tau = \widetilde{E}_T^2 \cdot \tau$, while, from section 5.3, we know that $\widetilde{E}_T^2 = 0$. Thus

$$i\omega \int_{\widehat{D}} \mu_r \, \widehat{H}_3^1 \, \mathrm{d}x = 0. \tag{96}$$

Finally, substituting (93) into (96) gives

$$C_m = -K_m \, I_0, \quad K_m := \left(\int_{\widehat{D}} \mu_r \, \mathrm{d}x \right)^{-1} \int_{\widehat{D}} \mu_r \, \partial_3\psi_m \, \mathrm{d}x. \tag{97}$$

To sum up, the fields E_3^1 and H_3^1 are given by

$$\left\{ \begin{aligned} & E_3^1 = \partial_3 V_0 \, (\varphi_e - \varphi_m) + V_0 \, \partial_3\varphi_e, \\ & H_3^1 = \mathrm{H}_3^s + \partial_3 I_0 \, (\psi_m - \psi_e) + I_0 \, (\partial_3\psi_m - K_m) \text{ with } K_m \text{ given by (97).} \end{aligned} \right. \tag{98}$$

5.5 Going back again to the conducting region

We are now in position to determine completely the fields $\widetilde{E}^2$ and $\widetilde{H}^0$. Since we already know that $\widetilde{H}^0_T \cdot n = 0$ (Section 5.1) , $\widetilde{H}^0_3 = 0$ and $\widetilde{E}^2_T = 0$ (Section 5.2), the remaining terms to be determined are $\widetilde{E}^2_3$ and $\widetilde{H}^0_T \cdot \tau$. As we are going to see, these will be the first non zero terms (thus the leading order terms) in the conducting region. Inspecting (51), one easily sees that two equations involving these two fields only come from the δ^{-2} of equation (51b) and the $O(1)$ term of (51c) (after substituting of (59) in (51)),

$$\sigma_{r,c}\, \widetilde{E}^2_3 + \partial_\eta \widetilde{H}^0_T \cdot \tau = 0, \quad i\omega\, \mu_{r,c}\, \widetilde{H}^0_T \cdot \tau + \partial_\eta \widetilde{E}^2_3 = 0. \tag{99}$$

Hence, eliminating $\widetilde{E}^2_3$, we see that $\widetilde{H}^0_T \cdot \tau$ satisfies

$$i\omega\, \mu_{r,c}\, \sigma_{r,c}\, \widetilde{H}^0_T \cdot \tau - \partial^2_\eta \widetilde{H}^0_T \cdot \tau = 0, \quad \text{in } \mathcal{Q}^0_\pm.$$

With the transmission condition (53b) and the condition at infinity (58) we have, since $\mu_{r,c}$ and $\sigma_{r,c}$ are constant in connected component of $\mathcal{Q}^0_\pm$

$$(\widetilde{H}^0_T \cdot \tau)(s, \eta, x_3) = (H^0_T \cdot \tau)(x_\pm(s), x_3)\, e^{-\alpha_c^\pm \sqrt{i\omega}\, \eta}, \quad \text{in } \mathcal{Q}^0_\pm \tag{100}$$

with

$$\alpha_c^\pm := \sqrt{\mu^\pm_{r,c}\, \sigma^\pm_{r,c}},$$

and where we use the principal determination of the complex square root in $\mathbb{C} \backslash \mathbb{R}^-$:

$$z = \rho\, e^{i\theta},\ \rho \geq 0,\ |\theta| < \pi\ \Rightarrow \sqrt{z} = \ \rho^{\frac{1}{2}}\, e^{i\theta/2}\ \text{ so that } \mathcal{R}e\, \sqrt{z} > 0,\ \ \forall\, z \in \mathbb{C} \setminus \mathbb{R}^-.$$

Then, we deduce from (99) that

$$\widetilde{E}^2_3(s, \eta, x_3) = \beta_c^\pm \sqrt{i\omega}\ (H^0_T \cdot \tau)(x_\pm(s), x_3)\, e^{-\alpha_c^\pm \sqrt{i\omega}\, \eta}, \quad \text{in } \mathcal{Q}^0_\pm, \tag{101}$$

with

$$\beta_c^\pm := \sqrt{\frac{\mu^\pm_{r,c}}{\sigma^\pm_{r,c}}}.$$

In the same manner that we could determine $\widetilde{H}^0_T \cdot n$ in Section 5.1, we can determine $\widetilde{H}^1_T \cdot n$ as well, which we shall use in the next section. More precisely, after substituting (59) into (51c), the δ-term gives

$$i\omega\, \mu_{r,c}\, \widetilde{H}^1_T - (\partial_s \widetilde{E}^2_3)\, n + (\partial_\eta \widetilde{E}^3_3)\, \tau = 0,$$

from which we deduce, even though $\widetilde{E}^3_3$ is still unknown, that

$$i\omega\, \mu_{r,c}\, \widetilde{H}^1_T \cdot n = \partial_s \widetilde{E}^2_3$$

and therefore, using (101), we obtain that in $\mathcal{Q}^0_\pm$,

$$\big(\mu^\pm_{r,c}\, \widetilde{H}^1_T \cdot n\big)(s, \eta, x_3) = \partial_s \left(\frac{\beta_c^\pm}{\sqrt{i\omega}}\, H^0_T(x_\pm(s), x_3) \cdot \tau(s) \right) e^{-\alpha_c^\pm \sqrt{i\omega}\, \eta}. \tag{102}$$

Remark 2 For sake of simplicity, we have assumed that the conductivity $\sigma_{r,c}$ and the permeability $\mu_{r,c}$ are constant in the conductors. If these coefficients are not constant then the expression of the electromagnetic fields are more complicated inside the conductors. More precisely,

- If the coefficients depend smoothly on the longitudinal variable s then (100) and (101) is valid (with $\alpha_c^\pm$ and $\beta_c^\pm$ depending on s). However, (102) is no longer valid (one term is actually missing), it should read

$$\big(\mu_{r,c}^\pm\, \widetilde{H}_T^1 \cdot n\big)(s,\eta,x_3) = \partial_s \left(\frac{\beta_c^\pm}{\sqrt{i\omega}}\, H_T^0(x_\pm(s),x_3)\cdot\tau(s)\right)\, e^{-\alpha_c^\pm \sqrt{i\omega}\,\eta}$$

$$-\,\beta_c^\pm\, \partial_s \alpha_c^\pm\, H_T^0(x_\pm(s),x_3)\cdot\tau(s)\; e^{-\alpha_c^\pm \sqrt{i\omega}\,\eta}. \quad (103)$$

- If the coefficients depend on the normal variable η then expression (100) is no longer valid and the solution of the boundary value problem (99) is much more complicated. Note however that such a dependence implies, in the physical variable $\mathbf{x}$, a sharp variation of the physical coefficients, it seems then natural that it deserves a dedicated treatment.

5.6 The first order transverse electromagnetic field in the dielectric

This is where we are going to determine the fields H_T^1 and E_T^1 that are the first terms in the electric that are affected by the boundary layers in the conductors. Substituting (56) into (44b) and (44d) and identifying terms in $O(1)$ we obtain, since $E_3^0 = H_3^0 = 0$ (Section 5.2)

$$(a)\;\; \mathrm{rot}\, H_T^1 = j_3^0, \quad (b)\;\; \mathrm{rot}\, E_T^1 = 0. \quad (104)$$

Proceeding similarly in the hidden divergence equation (47) and using (cf. (72)) $E_T^0 = V_0\, \nabla\varphi_e$, we get

$$(a)\;\; \mathrm{div}\, \varepsilon_{r,d}\, E_T^1 = -\frac{V_0}{i\,\omega}\, \mathrm{div}\, \sigma_{r,d}\, \nabla\varphi_e \quad (b)\;\; \mathrm{div}\, \mu_r\, H_T^1 = 0. \quad (105)$$

To complete (104) and (105) with boundary conditions, we first observe that, from the transmission conditions (53a) and $\widetilde{E}_T^1 = 0$ in $\mathcal{Q}_\pm^0$ (Section 5.1), we have

$$E_T^1 \cdot \tau = 0 \quad \text{on } \partial\widehat{D}. \quad (106)$$

Since, by (54b), $\mu_{r,d}\, H_T^1(x_\pm(s),x_3)\cdot n = \mu_{r,c}\, \widetilde{H}_T^1(s,0,x_3)\cdot n$ on $\Sigma_\pm$, we get from (102) for $\eta = 0$

$$\mu_{r,d} H_T^1 \cdot n = \partial_s \left(\frac{\beta_c^\pm}{\sqrt{i\omega}}\, H_T^0 \cdot \tau \right) = I_0\; \partial_s \left(\frac{\beta_c^\pm}{\sqrt{i\omega}}\, \widetilde{\nabla}\psi_m \times n \right) \quad \text{on } \partial\widehat{D}. \quad (107)$$

The resolution of (104, 105, 106, 107), seen as equations for (E_T^1, H_T^1), will be done in two steps. First we deduce representations of the electric and magnetic fields similar to (72), using Theorem 1. The main difference with Section 5.2 is that equations (104)(a), (105)(a) and (107) are non homogeneous. As it was already the case for the leading order fields, these representations will involve a electric voltage and current for which we will deduce 1D equation in a second step.

Structure of the electric and magnetic field. For E_T^1, in order to deal with the right hand side of (105)(a), it is natural to introduce a new potential $\varphi_e^c \in H_0^1(\widehat{D})$ (c for conductivity) , related to the conductivity $\sigma_{r,d}$ in the dielectric, solution of the problem

$$\operatorname{div} \varepsilon_{r,d} \nabla \varphi_e^c = -\operatorname{div} \sigma_{r,d} \nabla \varphi_e \quad \text{in } \widehat{D}, \quad \varphi_e^c = 0 \quad \text{on } \partial\widehat{D}. \tag{108}$$

Then, it is readily seen that $E_T^{1,*} := E_T^1 - \dfrac{V_0}{i\omega} \nabla \varphi_e^c$ satisfies

$$\operatorname{div} \varepsilon_r E_T^{1,*} = 0, \quad \operatorname{rot} E_T^{1,*} = 0, \quad \text{in } \widehat{D}, \quad E_T^{1,*} \times n = 0 \quad \text{on } \partial\widehat{D}.$$

Thus by Theorem 1(a), there exists $V_1(x_3) : \mathbb{R} \to \mathbb{C}$ (first order voltage) such that

$$E_T^1 = V_1 \nabla \varphi_e + \frac{V_0}{i\omega} \nabla \varphi_e^c. \tag{109}$$

For H_T^1, we have to introduce two additional potentials $\varphi_m^s \in H_0^1(\widehat{D})$ (s for source), in order to treat the (volumic) source r.h.s. of (104)(a) due to the source term j_3^0, and $\psi_m^b \in H^1(\widehat{D})$ (b for boundary) in order to treat the (boundary) r.h.s. of (107). These are defined by

$$\operatorname{rot} \mu_{r,d}^{-1} \,\mathbf{rot}\, \varphi_m^s = j_3^0 \quad \text{in } \widehat{D}, \quad \varphi_m^s = 0 \quad \text{on } \partial\widehat{D}, \tag{110}$$

and (see Remark 3 for the well-posedness of (111))

$$\operatorname{div} \mu_{r,d} \nabla \psi_m^b = 0 \text{ in } \widehat{D}, \quad \mu_{r,d} \nabla \psi_m^b \cdot n = g_m^b \text{ on } \partial\widehat{D}, \quad \int_{\widehat{D}} \mu_{r,d}\, \psi_m^b \,\mathrm{d}x = 0. \tag{111}$$

where we have set

$$g_m^b := \partial_s(\beta_c^{\pm} \widetilde{\nabla} \psi_m \times n) \quad \text{on } \partial\widehat{D}. \tag{112}$$

Then, it is readily seen that $H_T^{1,*} := H_T^1 - \dfrac{I_0}{\sqrt{i\omega}} \nabla \psi_m^b - \mu_r^{-1} \mathbf{rot}\, \varphi_m^s$ satisfies

$$\operatorname{div} \mu_r H_T^{1,*} = 0, \quad \operatorname{rot} H_T^{1,*} = 0, \quad \text{in } \widehat{D}, \quad H_T^{1,*} \cdot n = 0 \quad \text{on } \partial\widehat{D}.$$

Thus, from Theorem 1(b), there exists $I_1(x_3) : \mathbb{R} \to \mathbb{C}$ (first order current) so that

$$H_T^1 = I_1 \widetilde{\nabla} \psi_m + \frac{I_0}{\sqrt{i\omega}} \nabla \psi_m^b + \mu_r^{-1} \,\mathbf{rot}\, \varphi_m^s. \tag{113}$$

Remark 3 Note that, to obtain a well-posed problem, the r.h.s. g_m^b of (111) should at least belong to $H^{-1/2}(\partial\widehat{D})$ or equivalently that

$$\widetilde{\nabla} \psi_m \times n \in H^{1/2}(\partial\widehat{D}), \text{ i. e. using (33)}, \quad \mu_{r,d}^{-1} \nabla \varphi_m \cdot n \in H^{1/2}(\partial\widehat{D}). \tag{114}$$

This is is guaranteed if $\mu_{r,d}$ is Lipschitz-continuous in a tubular neighbourhood $\mathcal{T}(\partial\widehat{D})$ of $\partial\widehat{D}$, assumption we already did in in Remark 1 for ensuring (28). Indeed, it is sufficient to prove that the trace of φ_m belongs to $H^{3/2}(\partial\widehat{D})$. However, using standard regularity results for elliptic problems (see [20, 21]) applied to the problem (30) that defines φ_m, since the boundary $\partial\widehat{D} = \Sigma_+ \cup \Sigma_-$ has C^2 regularity and $\mu_{r,d}$ is Lipschitz-continuous in $\mathcal{T}(\partial\widehat{D})$, we deduce that $\varphi_m \in H^2(\mathcal{T}(\partial\widehat{D}))$ and we can conclude with the trace theorem.

Moreover, one has to check that g_m^b has mean value 0 along $\partial\widehat{D}$ (this corresponds the usual compatibility conditions for Neumann problems). If we assume that the trace of φ_m belongs to $C^1(\partial\widehat{D})$, this simply results from the fact the the integral along each of the close curves $\Sigma_\pm$ of the $s-$derivative of a C^1-function is necessarily 0. The general case follows by density of $C^1(\partial\widehat{D})$ in $H^{-1/2}(\partial\widehat{D})$.

Equations for the electric voltage and current. We proceed as in section 5.2. The δ-terms issued from equations (44a) and (44c) give

$$\begin{cases} i\omega\,\varepsilon_r\,E_T^1 + \sigma_{r,d}\,E_T^0 - \partial_3\,(\mathbf{e}_3 \times H_T^0) - \mathbf{rot}\;H_3^2 = \mathbf{j}_T^1, \\ \\ i\omega\,\mu_r\,H_T^1 + \partial_3\,(\mathbf{e}_3 \times E_T^1) + \mathbf{rot}\;E_3^2 = \mathbf{0}. \end{cases} \tag{115}$$

Similarly to what we did in Section 5.2, the equations for (V_1, I_1) will be obtained by taking (for each value of x_3) the scalar product in $L^2(\widehat{D})$ of the two equations of (73) with $\nabla\varphi_e$ and $\widetilde{\nabla}\psi_m$ respectively.

To do so we first observe that, since $\varphi_e^c \in H_0^1(\widehat{D})$ and φ_e is solution of (19)

$$\int_{\widehat{D}} \varepsilon_{r,d}\,\nabla\varphi_e \cdot \nabla\varphi_e^c\;\mathrm{d}x = 0. \tag{116}$$

In the same way, because $\psi_m^b \in H^1(\widehat{D})$ and ψ_m is solution of (30)

$$\int_{\widehat{D}} \mu_{r,d}\,\widetilde{\nabla}\psi_m \cdot \nabla\psi_m^b\;\mathrm{d}x = \int_{\widehat{D}\setminus\Gamma} \mu_{r,d}\,\nabla\psi_m \cdot \nabla\psi_m^b\;\mathrm{d}x = 0. \tag{117}$$

Finally, by Green's formula (18), since $\varphi_m^s = 0$ on $\partial\widehat{D}$ and $\mathrm{rot}\,\widetilde{\nabla}\psi_m = 0$ (see (24)),

$$\int \mathbf{rot}\,\varphi_m^s \cdot \widetilde{\nabla}\psi_m\;\mathrm{d}x = 0. \tag{118}$$

Using (109, 113) and (26), one computes that, thanks to (116, 117, 118),

$$\int_{\widehat{D}} \varepsilon_{r,d}\,E_T^1 \cdot \nabla\varphi_e\;\mathrm{d}x = C\;V_1, \quad \int_{\widehat{D}} \mu_r\,H_T^1 \cdot \widetilde{\nabla}\psi_m\;\mathrm{d}x = L\;I_1.$$

One then obtains, using also (72) and the definitions (26, 27) again,

$$\begin{cases} i\omega\,C\,V_1 + G\,V_0 + \partial_3 I_1 - \displaystyle\int_{\widehat{D}} \nabla\varphi_e \cdot \mathbf{rot}\;H_3^2\;\mathrm{d}x = J_1, \\ i\omega\,L\,I_1 + \partial_3 V_1 + \displaystyle\int_{\widehat{D}} \widetilde{\nabla}\psi_m \cdot \mathbf{rot}\;E_3^2\;\mathrm{d}x = 0, \end{cases} \tag{119}$$

where the scalar source current J_1 is given by

$$J_1 = \int_{\widehat{D}} \nabla\varphi_e \cdot \mathbf{j}_T^1\,\mathrm{d}x. \tag{120}$$

To conclude, we observe that, with Green's formula (18),

$$\int_{\widehat{D}} \nabla\varphi_e \cdot \mathbf{rot}\;H_3^2\;\mathrm{d}x = \int_{\partial\widehat{D}} \nabla\varphi_e \times n\,H_3^2\,\mathrm{d}s = 0$$

while, using Green's formula again and the transmission condition (53d), one has

$$\int_{\widehat{D}} \widetilde{\nabla}\psi_m \cdot \mathbf{rot}\, E_3^2 \,\mathrm{d}x = \int_{\partial\widehat{D}} \widetilde{\nabla}\psi_m \times n \cdot E_3^2 \,\mathrm{d}s = \int_{\partial\widehat{D}} \widetilde{\nabla}\psi_m \times n \cdot \widetilde{E}_3^2 \,\mathrm{d}s$$

so that, using the expression (101) of $\widetilde{E}_3^2$ for $\eta = 0$ and $H_T^0 = I_0\, \widetilde{\nabla}\psi_m$

$$\int_{\widehat{D}} \widetilde{\nabla}\psi_m \cdot \mathbf{rot}\, E_3^2 \,\mathrm{d}x = \sqrt{i\omega}\, R\, I_0$$

where the resistance coefficient R has been defined by (29). Finally we obtain

$$\begin{cases} i\omega\, C\, V_1 + G\, V_0 + \partial_3 I_1 = J_1, \\ \\ i\omega\, L\, I_1 + \sqrt{i\omega}\, R\, I_0 + \partial_3 V_1 = 0. \end{cases} \tag{121}$$

Remark 4 If the conductivity $\sigma_{r,c}$ and the permeability $\mu_{r,c}$ in the conductors depend smoothly on the longitudinal variable s, then (as consequence of (103)), the equation (107) must be substitute by

$$\mu_{r,d} H_T^1 \cdot n = \frac{I_0}{\sqrt{i\omega}} \partial_s \left(\beta_c^{\pm}\, \widetilde{\nabla}\psi_m \times n \right) - I_0\, (\partial_s \alpha_{\pm}) \beta_c^{\pm}\, \widetilde{\nabla}\psi_m \times n \quad \text{on } \partial\widehat{D}. \tag{122}$$

One can show that the expression electric field E_T^1 is not modified and that of the magnetic field H_T^1 is again given by (113) with however ψ_m^b defined by (111) with

$$g_m^b := \partial_s(\beta_c^{\pm} \widetilde{\nabla}\psi_m \times n) - \sqrt{i\omega}\, (\partial_s \alpha_{\pm})\, \beta_c^{\pm}\, \widetilde{\nabla}\psi_m \times n \quad \text{on } \partial\widehat{D}. \tag{123}$$

instead of expression given in (112). Finally equation (121) still holds and the expression of the resistance coefficient R is again given by (29) but with β_c now a function of s.

6 A first order effective model

In this section we construct an effective model for electromagnetic fields inside the dielectric. The analysis of section 5 suggests to define approximate electromagnetic fields $\mathbf{E}_a^\delta$ and $\mathbf{H}_a^\delta$ such that

$$\mathbf{E}_a^\delta = E^0 + \delta E^1 \quad \text{in } \widehat{\Omega}_d \quad \text{and} \quad \mathbf{H}_a^\delta = H^0 + \delta H^1 \quad \text{in } \widehat{\Omega}_d. \tag{124}$$

Formally these formulas provides approximations of the true electromagnetic field which are second order accurate in the dielectric part since formally, in $\widehat{\Omega}_d$,

$$E_a^\delta = E^\delta + O(\delta^2) \text{ with } E^\delta = O(1), \quad H_a^\delta = H^\delta + O(\delta^2) \text{ with } H^\delta = O(1).$$

By analogy with (124), we define approximate currents and voltages as

$$V_a^\delta := V^0 + \delta\, V^1, \quad I_a^\delta := I^0 + \delta\, I^1, \tag{125}$$

where (V^0, I^0) and (V^1, I^1) are defined by (75, 121). From (72, 109, 113), we observe that the approximate transverse electromagnetic fields in $\widehat{\Omega}_d$ satisfy

$$\begin{cases} E^\delta_{a,T}(x,x_3) = V^\delta_a(x_3)\nabla\varphi_e(x) + \delta \dfrac{V^\delta_a}{i\omega}\, \nabla\varphi^c_e + O(\delta^2), \\ H^\delta_{a,T}(x,x_3) = I^\delta_a(x_3)\widetilde{\nabla}\psi_m(x) + \delta\, \dfrac{I^\delta_a}{\sqrt{i\omega}}\, \nabla\psi^b_m + \delta\, \mu_r^{-1}\, \mathbf{rot}\, \varphi^s_m + O(\delta^2), \end{cases} \tag{126}$$

while the approximate longitudinal electromagnetic fields in $\widehat{\Omega}_d$ satisfy

$$\begin{cases} E^\delta_{a,3} = \delta\, \partial_3 V^\delta_a\, (\varphi_e - \varphi_m) + \delta\, V^\delta_a\, \partial_3\varphi_e + O(\delta^2), \\ \\ H^\delta_{a,3} = \delta\, \partial_3 I^\delta\, (\psi_m - \psi_e) + \delta\, I^\delta\, (\partial_3\psi_m - K_m) + \delta\, \mathrm{H}^s_3 + O(\delta^2). \end{cases} \tag{127}$$

with K_m given by (97). In other words, up to second order error terms in (126, 127) the approximate fields $\mathbf{E}^\delta_a$ and $\mathbf{H}^\delta_a$ can be evaluated with the help of the 1D functions V^δ_a and I^δ_a, computed from (V^0, I^0) and (V^1, I^1), respective solutions of the 1D models (75) and (121), through the formulas (125)-(127).

Our goal is now to provide (δ-dependent) 1D effective model, involving effective voltage and current V^δ_{ef} and I^δ_{ef} given in (34), allowing, by mimicking the formulas (126, 127), to compute effective electric and magnetic fields in the dielectric, $\mathbf{E}^\delta_{\text{ef}}$ and $\mathbf{H}^\delta_{\text{ef}}$, that will approximate the exact fields $\mathbf{E}^\delta$ and $\mathbf{H}^\delta$ with the same accuracy (namely $O(\delta^2)$) than the the approximate fields $\mathbf{E}^\delta_a$ and $\mathbf{H}^\delta_a$.

6.1 Voltage-Current 1D effective model

Performing the linear combination (75) + δ (121), one sees that the approximate voltage and current V^δ_a and I^δ_a defined by (125) satisfy

$$\begin{cases} i\omega\, C\, V^\delta_a + \delta\, G\, V^\delta_a + \partial_3 I^\delta_a = J^\delta + \delta^2\, G\, V^1, \\ \\ i\omega\, L\, I^\delta_a + \delta\, \sqrt{i\omega}\, R\, I^\delta_a + \partial_3 V^\delta_a = \delta^2\, \sqrt{i\omega}\, R\, I^1. \end{cases} \tag{128}$$

It is thus natural to propose an effective model, defining effective voltage and current V^δ_{ef} and I^δ_{ef}, by removing the $O(\delta^2)$ terms in (128), which gives

$$\begin{cases} i\omega\, C\, V^\delta_{\text{ef}} + \delta\, G\, V^\delta_{\text{ef}} + \partial_3 I^\delta_{\text{ef}} = J^\delta \\ \\ i\omega\, L\, I^\delta_{\text{ef}} + \delta\, \sqrt{i\omega}\, R\, I^\delta_{\text{ef}} + \partial_3 V^\delta_{\text{ef}} = 0. \end{cases} \tag{129}$$

In order to rewrite the system (129) in time domain, apart from the usual substitution $i\omega \mapsto \partial_t$, we have to identify the operator that corresponds, after Fourier transform in time, to the multiplication by the function $\sqrt{i\omega}$. It turns out, because we are considering causal functions, that we have the substitution $\sqrt{i\omega} \mapsto \partial_t^{1/2}$ where the half-derivative is defined in (35). Then, our effective 1D model in time domain is the effective telegrapher's equations (34).

Remark 5 The operator $\partial_t^{\frac{1}{2}}$ makes more difficult the time discretisation of (34), however there exist efficient methods, such as the one based on convolution quadrature discussed in [23] or [24]

6.2 Energy analysis of the Voltage-Current 1D effective model

In this section, we briefly address the question of the mathematical, and uniform stability in δ, of the system (34). The main issue is the obtention of a priori energy type estimates. The rest of the analysis being a matter of applying standard techniques for linear evolution problems will be omitted here.

To get such an estimate, we consider a smooth enough solution of (34), typically

$$\big(V_{\rm ef}^\delta, I_{\rm ef}^\delta\big) \in W^{1,\infty}(\mathbb{R}^+; L^2(\mathbb{R})) \cap L^\infty(\mathbb{R}^+; H^1(\mathbb{R})).$$

We multiply the first equation of (34) by V^δ and the second one by I^δ and, after integration in time between 0 and T (for any $T > 0$) and space over $\mathbb{R}$, we obtain

$$\mathcal{E}^\delta(t) + \int_0^t \int_{\mathbb{R}} G\,|V_{\rm ef}^\delta|^2 \, {\rm d}x_3 \, {\rm d}\tau + \int_{\mathbb{R}} R \int_0^t \big(\partial_t^{\frac12} I_{\rm ef}^\delta\big)\, I_{\rm ef}^\delta \, {\rm d}x_3 \, {\rm d}\tau = \int_0^t \int_{\mathbb{R}} J^\delta \, V_{\rm ef}^\delta \, {\rm d}x_3 \, {\rm d}\tau, \quad (130)$$

where the energy $\mathcal{E}^\delta(t)$ is defined by

$$\mathcal{E}^\delta(t) = \frac12 \int_{\mathbb{R}} C(x_3)\,|V_{\rm ef}^\delta(x_3,t)|^2 \, {\rm d}x_3 + \frac12 \int_{\mathbb{R}} L(x_3)\,|I_{\rm ef}^\delta(x_3,t)|^2 {\rm d}x_3,$$

and where we have used the fact that this energy vanishes at $t = 0$ because of the initial conditions.

Lemma 1 *Let $f \in W^{1,\infty}(\mathbb{R}^+)$, $f \neq 0$, $f(0) = 0$ and $T > 0$, then*

$$\int_0^T \big(\partial_t^{\frac12} f(t)\big)\, f(t) \; dt \; > \; 0. \quad (131)$$

Proof The result is essentially a straightforward consequence of Plancherel's theorem but the proof requires an approximation process.

Given $\epsilon > 0$, an approximation parameter devoted to tend to 0, we construct $f_{\epsilon,T} \in W^{1,\infty}(\mathbb{R}) \cap H^1(\mathbb{R})$ with support in $[0, T + \epsilon]$ such that

$$f_{\epsilon,T}(t) = f(t) \text{ for } 0 \le t \le T, \;\; f_{\epsilon,T}(t) = f(T)\,\Big(1 - \frac{t-T}{\epsilon}\Big) \text{ for } T \le t \le T+\epsilon,$$

Let $\widehat{f}_{\epsilon,T}(\omega)$ be the Fourier transform of $f_{\epsilon,T}(t)$, by definition, the Fourier transform of the half-derivative of $f_{\epsilon,T}$ is $\sqrt{i\omega}\,\widehat{f}_{\epsilon,T}(\omega)$.

Since $\omega \mapsto (1+\omega^2)\,|\widehat{f}_{\epsilon,T}(\omega)|^2$ is integrable over $\mathbb{R}$, so is $\omega \mapsto |\omega|\,|\widehat{f}_{\epsilon,T}(\omega)|^2$. By Plancherel's theorem, the half-derivative of $f_{\epsilon,T}$ belongs to $L^2(\mathbb{R})$ and

$$\int_{\mathbb{R}} \partial_t^{\frac12} f_{\epsilon,T}(t)\; f_{\epsilon,T}(t)\; dt = \mathcal{R}e \int_{\mathbb{R}} \sqrt{i\omega}\; |\widehat{f}_{\epsilon,T}(\omega)|^2 \; d\omega \; > \; 0,$$

since $\mathcal{R}e\sqrt{i\omega} > 0$ for $\omega \neq 0$. By causality of the Caputo's half-derivative,

$$\partial_t^{\frac12} f_{\epsilon,T}(t)\; f_{\epsilon,T}(t) = \partial_t^{\frac12} f(t)\; f(t) \text{ for } 0 \le t \le T$$

and is supported in $[0, T+\epsilon]$ by construction. Therefore, we have

$$\int_0^T \partial_t^{\frac{1}{2}} f(t)\, f(t)\, dt + \int_T^{T+\epsilon} \partial_t^{\frac{1}{2}} f_{\epsilon,T}(t)\, f_{\epsilon,T}(t) dt \geq 0,$$

and one can conclude by passing to the limit when ϵ tends to 0, using

$$\forall\, T \leq t \leq T+\epsilon, \quad |f_{\epsilon,T}(t)| \leq |f(T)|, \quad |\partial_t^{\frac{1}{2}} f_{\epsilon,T}(t)| \leq |\partial_t^{\frac{1}{2}} f(T)| + \frac{2}{\sqrt{\epsilon}}\, |f(T)|.$$

■

Using Lemma 1 in (130), we thus get, with $C_* := \inf_{x\in\mathbb{R}} C(x) > 0$,

$$\mathcal{E}(t) \leq \int_0^t \int_{\mathbb{R}} J^\delta\, V^\delta \mathrm{d}x_3\, \mathrm{d}t \leq \sqrt{\frac{2}{C_*}} \int_{\mathbb{R}} \|J^\delta(\cdot,t)\|_{L^2(\mathbb{R})}\, \mathcal{E}(t)^{\frac{1}{2}}(t) \mathrm{d}t,$$

and, after time integration again, we get the Growall's type estimate

$$\sup_{t\in[0,T]} \mathcal{E}^{\frac{1}{2}}(t) \leq \sqrt{\frac{2}{C_*}} \int_0^T \|J^\delta\|_{L^2(\mathbb{R})}\, \mathrm{d}t.$$

which corresponds to uniform estimates in δ of the solution $\left(V_{\text{ef}}^\delta, I_{\text{ef}}^\delta\right)$.

6.3 Reconstruction formulae for the 3D electromagnetic field in the dielectric

We know define the effective electromagnetic field $(\mathbf{E}_{\text{ef}}^\delta, \mathbf{V}_{\text{ef}}^\delta)$ from the effective voltage and current V_{ef}^δ and I_{ef}^δ based on the relationships (126) and (127) between the approximate fields and the approximate voltage and current. In the frequency domain, for the transverse fields, this leads to

$$\begin{cases} \mathbf{E}_T^\delta = V_{\text{ef}}^\delta\, \nabla\varphi_e + \delta\, \dfrac{V^\delta}{i\omega} \nabla\varphi_e^c, \\[2ex] \mathbf{H}_T^\delta = I_{\text{ef}}^\delta\, \widetilde{\nabla}\psi_m + \delta\, \dfrac{I^\delta}{\sqrt{i\omega}}\, \nabla\psi_m^b\, + \delta\, \mu_r^{-1}\, \mathbf{rot}\, \varphi_m^s, \end{cases} \tag{132}$$

where φ_e^c is defined in (108), ψ_m^b in (111) and φ_m^s in (110). Note that, by definition φ_e^c and ψ_m^b depend only on the geometry of the cable and of the physical parameters whereas φ_m^s depends linearly on the source term j_3^0, which shows that the longitudinal component of the source term has only a local quasi-static contribution: for every x_3 in which j_3^0 vanishes so does φ_j. To go back to the time domain from (132, 134), we observe that $1/(i\omega)$ corresponds (since we deal with functions that vanish at $t=0$) to the primitive in time and we write $1/(i\omega)^{\frac{1}{2}} = (i\omega)^{\frac{1}{2}}/i\omega$ in order to use the Caputo's half derivative we can easily go back to time domain. We get,

$$\begin{cases} \mathbf{E}_T^\delta = V_{\text{ef}}^\delta\, \nabla\varphi_e + \delta \left(\displaystyle\int_0^t V^\delta\, \mathrm{d}s\right)\, \nabla\varphi_e^c, \\[2ex] \mathbf{H}_T^\delta = I_{\text{ef}}^\delta\, \widetilde{\nabla}\psi_m + \delta\, \partial_t^{\frac{1}{2}} \left(\displaystyle\int_0^t I^\delta\, \mathrm{d}s\right)\, \nabla\psi_m^b\, + \delta\, \mu_r^{-1}\, \mathbf{rot}\, \varphi_m^s. \end{cases} \tag{133}$$

For the longitudinal fields, we propose the following reconstruction, which is valid in both frequency and time domains

$$\left\{\begin{aligned} &\mathbf{E}_3^\delta = \delta\, \partial_3 V_{\mathrm{ef}}^\delta\, (\varphi_e - \varphi_m) + \delta\, V_{\mathrm{ef}}^\delta\, \partial_3 \varphi_e, \\ &\mathbf{H}_3^\delta = \delta\, \mathrm{H}_3^{s,\delta} + \delta\, \partial_3 I_{\mathrm{ef}}^\delta\, (\psi_m - \psi_e) + \delta\, I_{\mathrm{ef}}^\delta\, (\partial_3 \psi_m - K_m) \end{aligned}\right. \tag{134}$$

where K_m is defined by (97) while $\mathrm{H}_3^{s,\delta}$ is defined in a similar way as in (92):

$$\left\{\begin{aligned} &\text{for each } x_3 \in \mathbb{R},\ \mathrm{H}_3^{s,\delta}(\cdot, x_3) \in H^1(\widehat{D}) \text{ is the scalar field uniquely by} \\ &\mathbf{rot}\, \mathrm{H}_3^{s,\delta} = \mathbf{j}_T^\delta - C^{-1} J^\delta\, \varepsilon_{r,d}\, \nabla \varphi_e \ \text{ in } \widehat{D} \quad \text{and} \quad \int_{\widehat{D}} \mu_r\, \mathrm{H}_3^{s,\delta}\, \mathrm{d}x = 0. \end{aligned}\right. \tag{135}$$

7 Effective parameters for circular symmetric cables

In this section we derive explicit formulae for the coefficient $RLCG$ defined by (26, 27, 29) in the case where, for each x_3 fixed the dielectric cross-section is an annulus with inner radius $\rho_- > 0$ and outer radius $\rho_+ > \rho_-$. Consequently, we introduce the usual polar coordinate (r, θ) and basis vector (e_r, e_θ) with $\theta \in [0, 2\pi)$. We further assume that the coefficients $\mu_{r,d} = \mu_{r,d}(r)$ and $\varepsilon_{r,d} = \varepsilon_{r,d}(r)$ depend only on the radial coordinate. Finally, the cut Γ is given, in polar coordinates by

$$\Gamma = [\rho_-, \rho_+] \times \{0\}.$$

Proposition 1 *With the assumptions above, we have*

$$L = \frac{1}{2\pi} \int_{\rho_-}^{\rho_+} \frac{\mu_{r,d}(r)}{r} dr, \quad C = 2\pi \left(\int_{\rho_-}^{\rho_+} \frac{\varepsilon_{r,d}^{-1}(r)}{r} dr \right)^{-1} \tag{136}$$

and

$$G = \frac{C^2}{4\pi^2} \int_0^{2\pi} \int_{\rho_-}^{\rho_+} \frac{\sigma_{r,d}(r,\theta) \varepsilon_{r,d}^{-2}(r)}{r} dr d\theta, \quad R = \frac{\beta_c^-}{2\pi\rho_-} + \frac{\beta_c^+}{2\pi\rho_+}. \tag{137}$$

Proof Firstly, we can remark that $\psi := (1 - \frac{\theta}{2\pi})$ satisfy for any radial field $\lambda(r)$

$$\left\{\begin{array}{ll} \operatorname{div} \lambda \nabla \psi = 0 & \text{in } \widehat{D} \setminus \Gamma, \\ \lambda \nabla \psi \cdot n = 0 & \text{on } \partial \widehat{D}, \\ [\psi]_\Gamma = 1, \quad [\lambda \nabla \psi \cdot n]_\Gamma = 0, & \text{across } \Gamma, \end{array}\right.$$

Consequently, ψ_m and ψ_e defined by (20, 21) and (31, 32) respectively are equal to ψ up to a constant. In particular,

$$\widetilde{\nabla} \psi_m = \widetilde{\nabla} \psi_e = \widetilde{\nabla} \psi = -\frac{1}{2\pi r} e_\theta, \quad \text{in } \widehat{D}. \tag{138}$$

Therefore, the definition of L given by (26) reads in polar coordinates

$$L = \int_0^{2\pi} \frac{\mu_{r,d}(r)}{(2\pi r)^2} r dr d\theta = \frac{1}{2\pi} \int_{\rho_-}^{\rho_+} \frac{\mu_{r,d}(r)}{r} dr.$$

To obtain explicit formulae for C and G we first rewrite their expressions in terms of the potential ψ_e. Using (26, 27) and (33)(a) one can easily show that

$$C = \left(\int_{\widehat{D}} \varepsilon_{r,d}^{-1} \, |\widetilde{\mathbf{rot}}\, \psi_e|^2 \, \mathrm{d}x \right)^{-1}, \quad G = C^2 \int_{\widehat{D}} \sigma_{r,d} \varepsilon_{r,d}^{-2} \, |\widetilde{\mathbf{rot}}\, \psi_e|^2 \, \mathrm{d}x, \tag{139}$$

then, substituting the value of $\widetilde{\nabla}\psi_e$ given by (138) we obtain the value of C and G as given in (136, 137). For the resistance coefficient, the value of $\widetilde{\nabla}\psi_m$ given by (138) and the formula (29) in polar coordinate gives the expected formula. ■

Observe that the expression of the resistance coefficient is, in this specific situation, independent of the physical parameters in the dielectric. This expression coincides with the one given in the classical engineering literature (see [17,14,15, 4]). Moreover, if we assume that $\mu_{r,d}, \varepsilon_{r,d}$ and $\sigma_{r,d}$ are constant, we recover the usual formulae (given for instance in [14,15])

$$L = \mu_{r,d} \, \mathfrak{g}, \quad C = \varepsilon_{r,d} \, \mathfrak{g}^{-1} \quad \text{and} \quad G = \sigma_{r,d} \, \mathfrak{g}^{-1} \quad \text{with} \quad \mathfrak{g} = \frac{\ln(\rho_+/\rho_-)}{2\pi}$$

Thus, showing that the expression of the coefficients $RLCG$ given in (26,27,29) are a generalisation of the one given in the engineering literature in the homogeneous case.

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